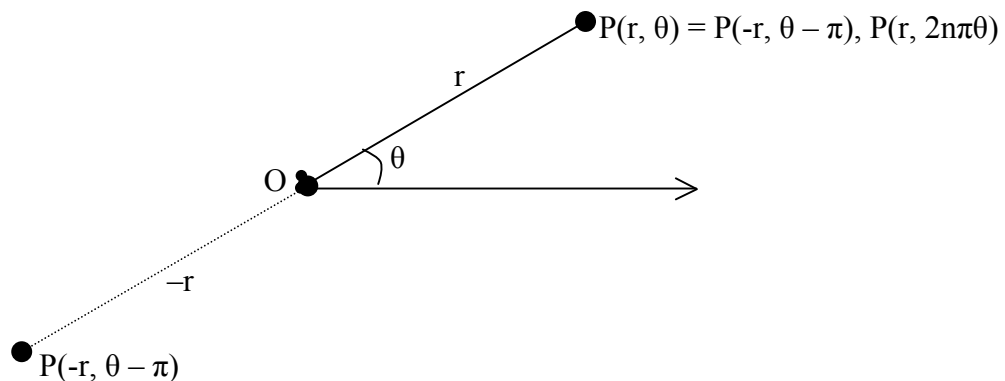## §9.4 Polar Coordinates

**Polar coordinates** uses *distance* and *direction* to specify a location in a plane.

The **origin** in a polar system is a fixed point from which a ray, O, is drawn and we call the ray the **polar axis** (what we would have called the initial side in previous study). The distance from O to P (P is the endpoint of a ray rotated through an angle θ; what we would have called a terminal point in previous study). The distance from O to P is called "r", the radius. P is assigned an ordered pair determined by "r" and θ, and labeled with the **polar coordinates** P(r, θ).

In   P(r, θ)
     r = distance from O to P
     θ = Angle that OP makes with the polar axis

Note: A negative r lies on the opposite end of the ray with angle θ

$P(r, θ) = P(-r, θ – π), P(r, 2nπθ)$

$P(-r, θ – π)$

Example: Plot $P(2, {}^π/_6)$ & $P(3, {}^{2π}/_3)$ and then give 2 other polar coordinates for these points.
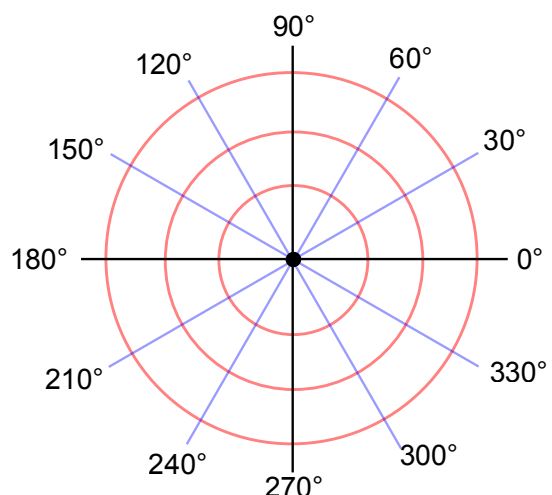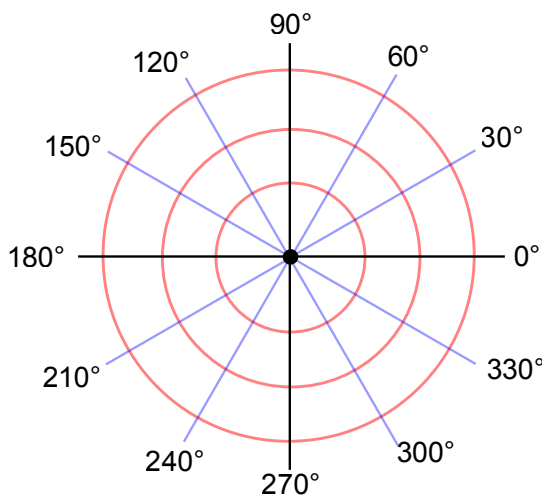
The relationship between Cartesian & Polar coordinates is the relation that we saw in right triangles on the unit circle. The following is what we need to know to change **polar to Cartesian coordinates**.

$$\frac{x}{r} = \cos\theta \quad \text{so} \quad x = r\cos\theta$$

$$\frac{y}{r} = \sin\theta \quad \text{so} \quad y = r\sin\theta$$

Thus,   P(1, $^\pi/_6$) is $x = 1\cos{^\pi/_6} = 1 \cdot {^{\sqrt3}/_2}$
                                     $y = 1\sin{^\pi/_6} = 1 \cdot {^1/_2}$ } in the Cartesian system

            P(2, $^\pi/_6$) is $x = 1\cos{^\pi/_6} = 2 \cdot {^{\sqrt3}/_2} = \sqrt3$
                                     $y = 1\sin{^\pi/_6} = 2 \cdot {^1/_2} = 1$ } in the Cartesian system

   Example: Find P(-3, $^{2\pi}/_3$) in the Cartesian system

*Note: We saw that this point lies in QIV earlier and that is what the Cartesian coordinates show.

Now, we will go from **Cartesian to polar coordinates.** We will need to know the following to help us in this process.

$$r^2 = x^2 + y^2 \quad \& \quad \tan\theta = \frac{y}{x}$$

   Example: What are the polar coordinates of (-√6, -√2)
                    *#38 p. 546 of Stewart's *Precalculus* Ed. 6

***Note:** You must be careful because the quadrant information must still conform, hence why we don't use*
$^\pi/_6$ *and instead use* $^{7\pi}/_6$

The Polar Coordinate System consists of concentric circles with their center at O, the origin. The circles represent r, the lines emanating from O represent rays with varying angles θ made with the polar axis. Here is one example of a polar system.
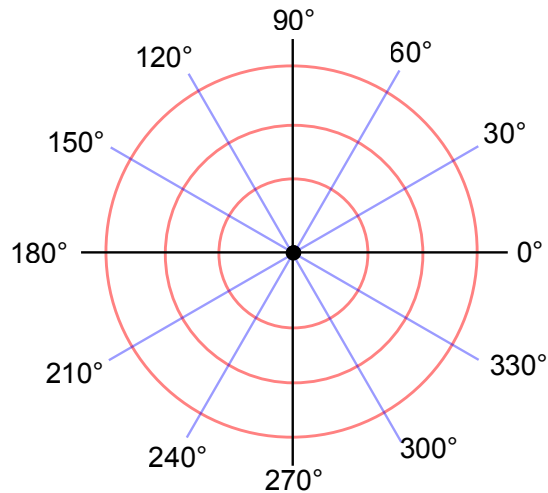
Some places that you might locate polar graph paper for your use are:
http://www.marquis-soft.com/graphpapereng.htm
http://mathematicshelpcentral.com/graph_paper.htm

Here's what you need to learn – just as in Algebra, certain functions have certain graphic representations and certain features that lead to key graphic features. We are going to learn some of the basic polar equations and their graphic representations. Here is what you need to know, much of which you will find in the supplementary text book by Stewart on p. 553 of Ed 6:

Circles	$r = a$	(center at zero)
	$r = 2a \sin \theta$	(center at $\mid a \mid$, $(a, {}^{\pi}/_2)$)
	$r = 2a \cos \theta$	(center at $(a, 0)$)
Spiral	$r = a\theta$	
Straight Lines thru Origin	$\theta = a$	
Cardioids	$r = a(1 \pm \cos \theta)$	
	$r = a(1 \pm \sin \theta)$	

Roses **n-leaved when n is odd**
2n-leaved when n is even $\Big\}$ $r = a \sin n\theta$ or $r = a \cos n\theta$

Limacon $r = a \pm b \cos \theta$ **If a < b then there is a loop, if a > b then dimpled**
$r = a \pm b \sin \theta$ **If a = b then it is a Cardioid**

Lemniscates $r^2 = a^2 \sin 2\theta$ symmetric about y = x
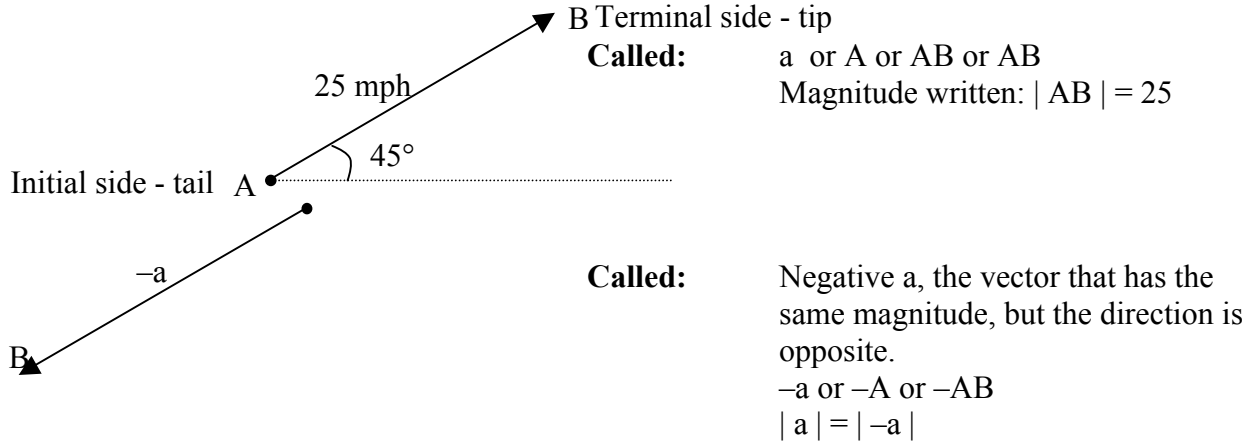 $r^2 = a^2 \cos 2\theta$ symmetric about the x-axis

You need to go over the examples in the supplemental Stewart text on p. 548-551 of Ed 6 paying attention to:

1)   Cartesian Graph (if even possible)
2)   One to one correspondence of the Cartesian & polar coordinates in Example 1 through 6
3)   Link between §8.1 & 8.2 in changing between polar & Cartesian coordinates. Note that an easy change seems to show 1:1 correspondence in graphs.
4)   Pay attention to the equations & picture lines as this is probably one of the places that I will test you.
          Ex. $r = 2 + 2 \cos \theta$ What would the general shape of the graph be?
5)   Symmetry (we will discuss this next)
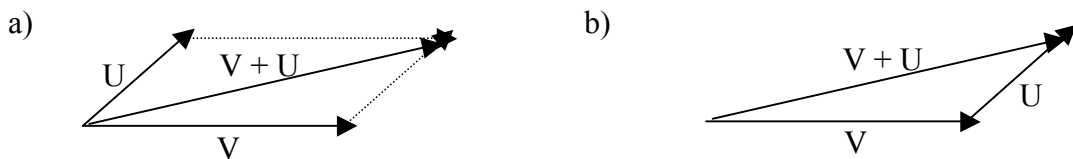
§9.5 Vectors

A <u>vector</u> involves *2 measures*, one of *direction* and one of *magnitude*. We can represent <u>vector quantities</u> with a directed line segment in which the length represents the magnitude and the angle represents the direction. Vectors, like line segments, can be written using 2 capital letters or a single lower or upper case letter. Because vectors are directional, when 2 capitals are used, the 1st represents the initial point and the 2nd the terminal point. The magnitude is the absolute value of the vector's magnitude. When a vector's tail is located at the origin it is called a <u>position vector</u>.

Example:

B Terminal side - tip

25 mph

45°

Initial side - tail   A

−a

B

Called: a or A or AB or AB
Magnitude written: | AB | = 25

Called: Negative a, the vector that has the same magnitude, but the direction is opposite.
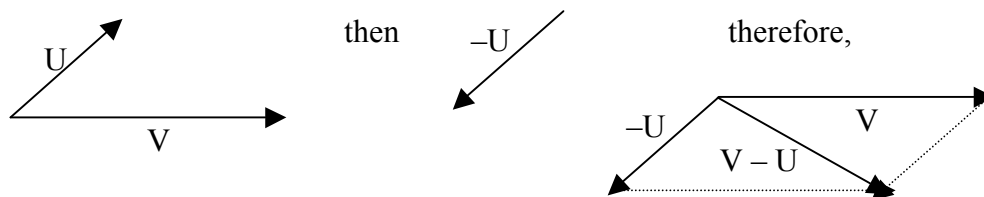−a or −A or −AB
| a | = | −a |

Adding vectors can be thought of as solving the long side of the triangle formed when 2 vectors are placed initial to terminal side or finding the diagonal of a parallelogram formed by placing initial sides together.

Example: V + U

a)

U

V + U

V

b)

V + U

U

V

The <u>difference</u> is found by adding the opposite of a vector.

Example: V − U

U

V

then

−U

therefore,

−U

V

V − U

Finally, we can make some sense of vectors if we put them in our familiar coordinate system. When a vector is placed with its initial point at the origin it is called a <u>position vector</u> with its endpoint at (a, b). We write a position vector u, in the following way:

$$u = <a, b>$$

where a & b are the horizontal and vertical components of the vector, respectively. The positive angle between the x-axis and the vector is the <u>direction angle</u>.

The Position Vector: **u = < a, b >**
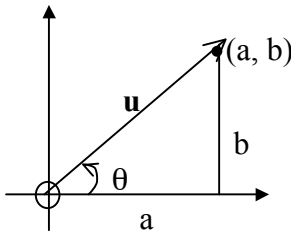
The Magnitude of **u**: $|u| = \sqrt{a^2 + b^2}$

$\tan \theta = \dfrac{b}{a}$, $a \neq 0$

$a = |u| \cos \theta$ **&** $b = |u| \sin \theta$

Note: Since the values of a and b are given as the magnitude times the sine or cosine values, another way of looking at the position vector is $< |u| \cos \theta, |u| \sin \theta >$

*Just think of all the coordinates that we've done that have been proportional to the unit circle. In the unit circle (cos θ, sin θ).

Example: Find the magnitude and direction of **u** = <-5, 4>

Now, more practice with the horizontal and vertical components.

Example: If **v** has | **v** | = 14.5 and θ = 220°, find the horizontal and vertical components of the position vector **v**

Example: Write each vector in terms of < a, b >
a)   **u**   if | **u** | = 8 **&** θ = 135°   b)   **v**   if | **v** | = 4 **&** θ = 270°

c)   **w**   if | **w** | = 10 **&** θ = 340°

Now let's practice adding vectors. Since we deal with parallelograms when adding and subtracting vectors, you should know the following facts about parallelograms:

        1)   A parallelogram is a quadrilateral with 2 sets of parallel sides

        2)   Opposites sides and angles are equal

        3)   Adjacent angles are supplementary

        4)   The diagonals bisect each other, but don't necessarily bisect the angles

Example: The vectors **u** & **v** are initial point to initial point. If | **u** | = 32 & | **v** | = 48 and the angle between them is 76°, find the magnitude of the resultant vector, **w**.

Using geometry, the following properties of operations with vectors can easily be shown. The following is a summary of what we need to know.

        1)   <a, b> + <c, d> = < a + c, b + d>
                Add "a" components & add "b" components

        2)   k • <a, b> = <k•a, k•b>
                Multiplying by a scalar, multiplies each component by the scalar.

        3)   If a = <a, b> **&** –a = <–a, –b>
                then <a, b> – <c, d> = <a, b> + –<c, d>
                = <a, b> + <–c, –d> = <a – c, b – d>
                From 1 & 2 above subtraction is the addition of an opposite.

Example: Let **u** = <6, -3> and **v** = <–14, 8>. Find
a)   **u** + **v**   b) $^{-1}/_2$ **v**   c)   5**u** + 2**v**

A <u>unit vector</u> has magnitude 1. We have 2 important unit vectors i & j (not to be confused with the i & j of complex numbers)

$$i = <1, 0> \qquad \& \qquad j = <0, 1>$$

Every vector <a, b> can be expressed in terms of i & j in the following manner:

$$<a, b> = a\,i + b\,j$$

since this means a< 1, 0 > + b< 0, 1 >

which is equivalent to < a, 0 > + < 0, b >

which is equivalent to < a + 0 , 0 + b > or < a, b >

The unit vector is important in applied sciences such as physics.

Example: Give the position vector for $4i + -7j$

Example: Find   a) 2**a**   b)   2**a** + 3**b**   c)   **b** − 3**a**   for
$$\mathbf{a} = -i + 2j \ \& \ \mathbf{b} = i - j$$
Write it in terms of both the position vector and as an expression in terms of i & j

§9.2 Dot Products – Supplementary Stewart text
Next, we will study another important application of vectors, the <u>dot product</u>. This is used to determine the angle between 2 vectors. It can help solve physics problems and derive geometric theorems.

Dot Product

$$\mathbf{u} \bullet \mathbf{v} = a \bullet c + b \bullet d$$

The dot product of vectors **u** & **v** is found by multiplying their
"a" & "b" components and summing the products.

Properties of Dot Products

1) $u \cdot v = v \cdot u$
   Dot products are commutative

2) $u \cdot (v + w) = u \cdot v + u \cdot w$
   Dot products are distributive

3) $(u + v) \cdot w = u \cdot w + v \cdot w$
   Dot products are commutative and distributive

4) $(ku) \cdot v = k(u \cdot v) = u \cdot (kv)$
   Multiplication by a scalar, is associative

5) $0 \cdot u = 0$
   Multiplication property of zero

6) $u \cdot u = |u|^2$
   Dot product of a vector with itself yields the magnitude squared

The angle between 2 vectors is very important, as we have seen (as in $u + v$ or $u - v$). Thus the next interpretation of the dot product is quite nice. Dugopoloski gives no explanation of this sort, but it is worth the time to look these over.

Geometric Interpretation of the Dot Product

When the angle θ between **u** & **v** is in the interval [0, 180°], then

$$u \cdot v = |\mathbf{u}||\mathbf{v}| \cos \theta$$

Example: To the nearest 10th of a degree, find the angle between
$\mathbf{u} = <5, -12>$ & $\mathbf{v} = <4, 3>$

Example: Find the angle between the vectors $2\mathbf{i} + \mathbf{j}$ and $-3\mathbf{i} + \mathbf{j}$

One last note before we close. When 2 vectors are at right angles they are called <u>orthogonal vectors</u>. The dot product of orthogonal vectors is zero (since the cosine of 90° is zero).

Example: Find the dot product of **u** = < 1 , 1 > & **v** = < −1 , 1 >. What does this mean?

Example: Is the following pair of vectors orthogonal? $-4\boldsymbol{i} + 3\boldsymbol{j}$ & $8\boldsymbol{i} - 6\boldsymbol{j}$

Applications of Vectors
See Evergreen Trig page and look up
Incline Plane Problem
Work Problem
Component Work Problem

Resultant Force happens when two forces are acting on a single object with an angle between the application of each force. The resultant force can be described as the diagonal of the parallelogram created by the two forces.

V + U
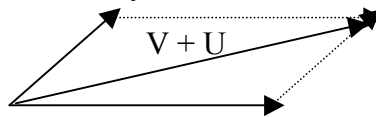
Here U & V are the two forces and V + U is representing the resultant force

Method 1 of Finding Resultant Force & Angle
1) Law of Cosines is used to find the magnitude of the resultant
2) Law of Sines is used to find the angle of the resulting force

Method 2 of Finding Resultant Force & Angle
1)   The component forms representing each vector must be found & then summed to
        find the component form of the resultant vector & finally the magnitude of that
        resultant vector can be calculated
2)   The angle of the resulting force can be found by $\tan^{-1} (^b/_a)$

Example: **6.** *Magnitude of a Force*   The resultant of a 15-lb force and an-
                other force has a magnitude of 9.8 lb at an angle of 31° with
                the 15-lb force. Find the magnitude of the other force and the
                angle between the other force and the resultant.

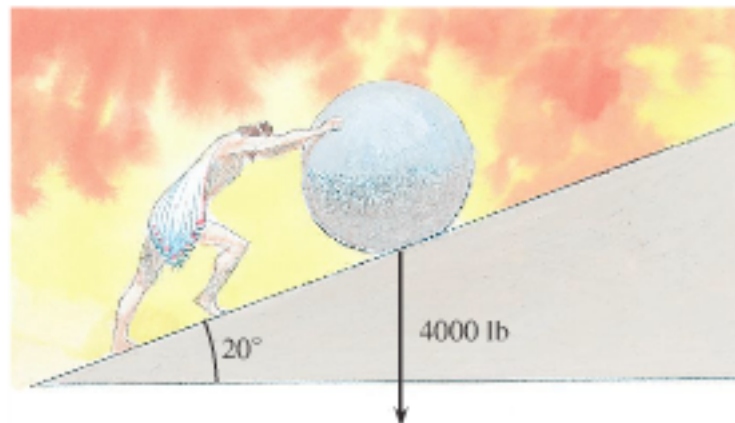   **#6 p. 296 *Trigonometry,* Dugopolski, 3rd Edition**

An **inclined plane** problem tells us about the force that is required to move an object up a
ramp. To calculate the forces at work, we must take into account that the resultant force
works *perpendicularly to the inclined plane* and therefore the component force up the
ramp is the force with which we must push and the other is the force with which gravity
is pulling.

1)   The weight of the object is the force with which gravity is pulling downward.
2)   The sine of angle the inclined plane makes with the ground is equivalent to the
        force divided by the weight.
$$\sin \theta = \frac{\text{force}}{\text{weight}}$$

*Remember that you may be asked to find the force, the weight or the angle of incline in
one of these type of problems.

Example: 10. *Rock and Roll*   In Roman mythology, Sisyphus, king of Corinth, revealed a secret of Zeus and thus incurred the god's wrath. As punishment, Zeus banished him to Hades where he was doomed for eternity to roll a rock uphill, only to have it roll back on him. If Sisyphus stands in front of a 4000-lb spherical rock on a 20° incline, as shown in the figure, then what force applied in the direction of the incline would keep the rock from rolling down the incline?

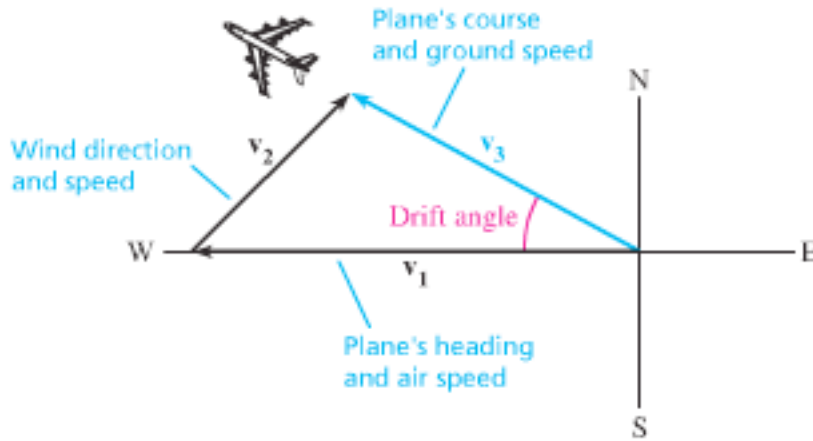#10 p. 297 *Trigonometry,* Dugopolski, 3rd Edition

Example: **12.** *Weight of a Ball*   A solid steel ball is placed on a 10° incline. If a force of 3.2 lb in the direction of the incline is required to keep the ball in place, then what is the weight of the ball?

#12 p. 297 *Trigonometry,* Dugopolski, 3rd edition

Example: 14. *Sisy's Slope*   If Sisyphus exerts a 500-lb force in rolling his 4000-lb spherical boulder uphill, then what is the angle of inclination of the hill?

#14 p. 297 *Trigonometry,* Dugopolski 3rd Edition

Navigation problems are a third type of problem that we will be dealing with in this sections. Below is the diagram that describes the elements in a navigation problem.

p. 293 *Trigonometry,* Dugopolski, 3rd Edition

Recall that **bearing** is the degrees from North in a clockwise direction. Heading due east is therefore a bearing of 90º. We will compute the drift angle with respect to the ground (east-west horizontal line) and then we will have to compute the bearing based upon that drift angle (add/subtract from 90º or 270º).

The **course & ground speed** will be a vector in standard position.

The heading and airspeed will give us angles and magnitude of a side of a triangle & the **wind direction and speed** (could be current speed/direction alternately) will give us another side of the triangle. *Note* that wind speed is given in terms of the direction from which is comes and toward which it blows.

Methodology of Solving Navigation Problems
1)   We will use the Law of Cosines to solve the airspeed.
2)   We will use the Law of Sines to solve the drift angle
3)   Compensate the course appropriately using the original heading & drift angle

Example: 18. *Superlight*   A superlight is flying northwest at 75 mph. The wind is from the south at 40 mph. Find the bearing for the course and the ground speed of the superlight.

#18 p. 297 *Trigonometry,* Dugopolski, 3rd Edition