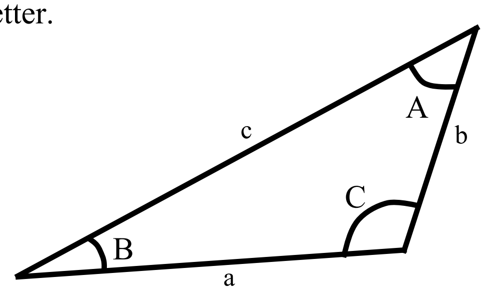
§9.3 The Law of Sines

First we need the definition for an **oblique triangle**. This is nothing but a triangle that is not a right triangle. In other words, all angles in the triangle are not of a measure of 90º. These are the type of triangle that are of interest to us in the application of the **Law of Sines** and **Law of Cosines**.

Next, let's review the naming convention of triangles (from your study of Geometry, hopefully). The angles are name with capital letters. The sides opposite an angle are named with the comparable lower case letter.



Now, we need to refresh our memory of some axioms from Geometry.

Congruency Axioms (Methods to establish congruency.)

- **SAS**: Side Angle Side — 2 sides known w/ included angle
- **ASA**: Angle Side Angle — 2 angles known w/ included or non-included side (AAS non-included side)
- **SSS**: Side Side Side — 3 sides known

Note: *We will always need at least 1 side to have congruency.*

It is with these axioms that we will use the **Law of Sines** and **Law of Cosines** that are the methods for solving oblique triangles.

There are a total of 4 cases for using the **Law of Sines** or **Law of Cosines**. They are as follows:

CASE 1: When we know one side and 2 angles. (SAA and ASA)

CASE 2: When two sides and one angle are known. (SSA—Ambiguous Case and SAS)

***CASE 3:** When two sides and 1 included angle are known. (SAS)

***CASE 4:** When three sides are known. (SSS)

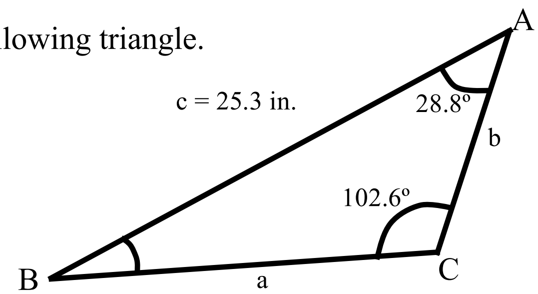
*We will see these cases in the next section using the Law of Cosines.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \textbf{OR} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Where A, B & C are angles in an oblique triangle and a, b & c are the sides opposite the angles.

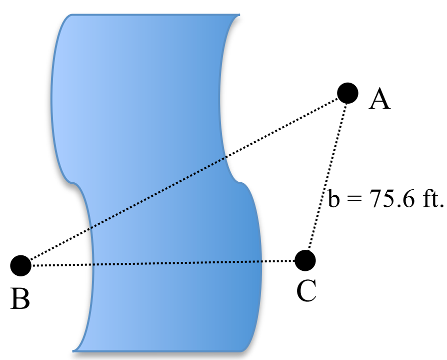
The above stated is called the **Law of Sines**. When solving a triangle using the Law of Sines, any two of the equivalent ratios will do. I recommend that you choose the relation that puts your unknown in the numerator for your own sanity.

Example: Solve the following triangle.



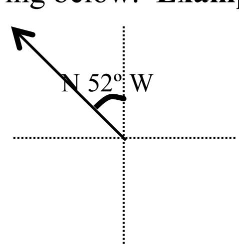
Is this a SAA, ASA or SSA or SAS? Do you need to worry about ambiguity?

Example: Seinfeld wants to measure the distance across the Hudson River (B to C). If he stands at point A he is 75.6 ft. from point C, on the Hudson's near bank. The angle of BCA is 117.2º and BAC is 28.8º. Refer to the diagram below.



What is this case? Is there any ambiguity?

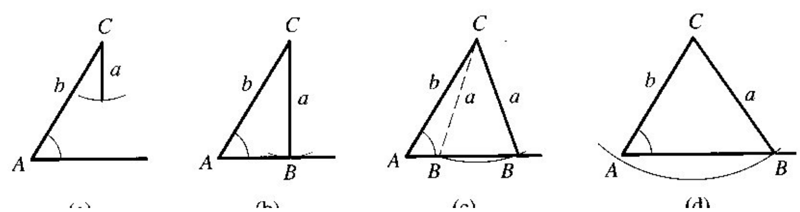
Bearing is something that is applied in some of our problems. Bearing is the angle from an afore mentioned pole in a horizontal direction. For example the following bearing can be interpreted in the drawing below. **Example:** N 52º W



Example: The bearing of a lighthouse from a ship is N 52º W. After sailing 5.8 km due south the bearing from the ship to the lighthouse is N 23º W Find the distance of the ship from the lighthouse in each location.

Again, what case is this? Is there any ambiguity?

Now we will talk about the ambiguous case of the Law of Sines. When we know SSA, there are 4 possibilities – there may be **no** such triangle as shown in the 1st picture below, a **right** triangle (the 2nd picture), **two** such triangles (the 3rd picture) or **one** unique triangle. I'm just going to take a scan of the possibilities and put them in here because the time it would take to draw them would not be worth my while. You can find further discussion on p. 503 of Ed 5 & p. 471 of Ed 6.



Note: *One of the checks to see if we have a second possible solution is to see if sin A < 1. If it is then we need to check and can eliminate only if the sum of the angles is > 180º. Likewise if sin A ≥ 1 there is no solution.*

Here is a summary of information that may be helpful in solving the Ambiguous Case of Sines.

Facts in Applying The Law of Sines

1) For any ∠ in a Δ $\sin\theta$ is in (0, 1].
2) When $\sin\theta = 1$, $\theta = 90^\circ$ and the triangle must be a right Δ. If another angle is ≥ 90º then it is an ambiguous case.
3) $\sin\theta = \sin(180 - \theta)$; In other words, supplementary angles have the same sines
   It is this equality that gives us the 2 possible triangles based on the same sine
4) Smallest Angle opposite shortest side
   Medium Angle opposite medium side
   Largest Angle opposite largest side
   In triangles that aren't isosceles or equilateral

Example: Solve ΔABC if a = 17.9cm, c = 13.2 cm & C = 75.5º

***Note**: The sin A is larger than one so this is not possible.*

Example: Solve ΔABC if A = 61.4º, a = 35.5 cm & b = 39.2cm

***Note:** Note that Given < B and in this case you should check 180º – B as a second solution.*

Example: Solve ΔABC, given B = 68.7º, b = 25.4 in.& a = 19.6 in.

***Note:** Note Given > A & checking further you will see that the supplementary angle to A + B will be > 180 so a second angle can't exist.*

Example: Explain why no triangle ABC exists for the following
B = 93º, b = 42 cm and c = 48 cm

***Note:** See the last of my 4 notes above.*

In Summary Proceeding w/ the Ambiguous Case

1) If the sine of the unknown angle is > 1 then no triangle exists

2) If sine of the unknown angle is = 1 then exactly one triangle exists and it is a right triangle.

3) If the sine is between 0 & 1 then 1 or 2 triangles exist
   a) Find the angle for the 1^{st} triangle
   b) Find the supplement of the angle. If original angle + supplement is < 180 then a second triangle exists and you should find it.

We can find the area of a triangle using Heron's Formula, which is based on the Law of Sines. I'm going to have us solve the following problem to find the area of the triangle given.

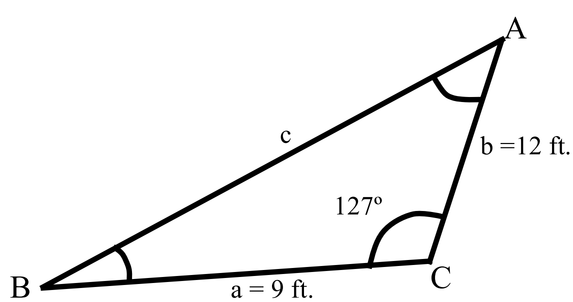
Heron's Formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s = {}^{1}/_{2}(a + b + c)$ & a, b &c are the side lengths

Note*: This works for SAS triangles only. If the included angle is 90º this is simply our familiar area formula since sin 90º is 1 and the two sides surrounding it are then the base and the height.*

Example: Use Heron's formula to find the area of the triangle shown below.



Example: Find the area of ΔABC if $B = 58\ {}^{1}/_{6}{}^{\circ}$, $a = 32.5$ cm and $C = 73\ {}^{1}/_{2}{}^{\circ}$

§10.1 Basic Identities

The first goal is to verify and rewrite trigonometric identities.

Guidelines as Outlines by Stewart, Redlin & Watson (p. 530 ed 5 & p. ed 6)

1) Start with one side (usually the more complicated)
   a) Pick one side & write it down
   b) Goal is to make it into the other side using identities
2) Use known identitites
   a) Use algebra
      i) Recall a perfect square trinomial: $a^2 \pm 2ab + b^2$ factors as $(a \pm b)^2$
         Ex. $9x^2 + 18x + 1 = (3x + 1)^2$
         and
         $4x^2 - 20x + 25 = (2x - 5)^2$
      ii) Finding an LCD & Building Higher Terms
         Ex. ${}^{1}/_{x} + {}^{1}/_{y} = {}^{y}/_{xy} + {}^{x}/_{xy} = {}^{(x + y)}/_{xy}$
      iii) Multiplying Conjugates yields the difference of squares
         Ex. $(2 + x)(2 - x) = 4 - x^2$
   b) Select identities that will bring your complicated side to a more simplistic end (headed toward the desired outcome)
3) Convert to sines & cosines
   a) Use reciprocal & quotient identities when possible
   b) Resort to Pythagorean identities when necessary

***Note:** Your book warns of trying to transform something that is not equivalent at face value by doing the same thing to both sides. This is a skill that you are familiar with and may lapse into out of habit. Be forewarned! The book's example was sin x = -sin x proven by squaring both sides. If a process is not invertible it should not be used – remember that squaring is not invertible unless the domain is restricted.*

Example: Verify that (cot x)(sin x)(sec x) = 1.
Hint: Put in terms of sin & cos

Example: Verify that $\cot^2 \theta(\tan^2 + 1) = \csc^2 \theta$ is an identity.
Hint: Use Pythagorean id & rewrite in terms of sin & cos

Example: Verify that $\dfrac{\tan^2 s}{\sec^2 s} = (1 + \cos s)(1 - \cos s)$

Hint: Focus on left to restate sec & Pythagorean Id to restate tan.

Example: Verify that $\dfrac{\sec s + \tan s}{\sin s} = \dfrac{\csc s}{\sec s - \tan s}$ is an identity

Hint: Focus on right. Use conjugate with fundamental theorem of fractions. Simplify & use another Pythagorean ID

Now let's practice using a different technique. Instead of focusing on just one side we'll **manipulate both sides**. We'll start with a second look at the third example above.

Example: Verify that $\dfrac{\tan^2 s}{\sec^2 s} = (1 + \cos s)(1 - \cos s)$

Hint: Rewrite left using sin & cos. Rewrite right by mult. out & using Pythagorean Id.

Example: Verify that $\dfrac{\cot \theta - \csc \theta}{\cot \theta + \csc \theta} = \dfrac{1 - 2\cos \theta + \cos^2 \theta}{-\sin^2 \theta}$

Hint: Right: Factor, Pythagorean Id, Factor

Left: Since csc & cot have sin in common, so mult by sin & simplify

Even-Odd Identities

$\sin(-\theta) = -\sin(\theta)$
$\cos(-\theta) = \cos(\theta)$
$\text{Tan}(-\theta) = -\tan(\theta)$

Cofunction Identities

$\sin A = \cos(90 - A)°$ **or** $\cos A = \sin(90 - A)°$
$\tan A = \cot(90 - A)°$ **or** $\cot A = (90 - A)°$
$\sec A = \csc(90 - A)°$ **or** $\csc A = \sec(90 - A)°$
*Note: $90 = {}^{\pi}/_{2}$ which is what your book uses

Example: Write $\cos(90 + \theta)$ as a single trig function of θ

Hint: Use the fact that $-(-\theta)$ is $+\theta$ and then use even/odd

Example: Write as a single trig function with a positive angle.

a) #13 p. 690 Adamson, Cox, Wilson & O'Bryan $\cos(-\theta) \div \sin(-\theta)$

b) #15 p. 690 Adamson, Cox, Wilson & O'Bryan $\cot(\theta)\csc(-\theta)\sec(-\theta)$

The Wilson, Cox, O'Bryan & Adamson book is really after the use of basic trig identities to solve trig equations. I don't believe that we've really explored this process through this text, so let's look at how other texts build this process by walking through solving trig equations from simple to more complex.

Recall conditional linear equations in Algebra had 1 solution. We are going to study conditional trig equations first. This means that some values will solve them and some will not.

<u>Method 1: Linear Methods</u>

1) Employ algebra to isolate the trig function
2) Use trig identities as needed
3) Use definitions as needed
4) When using the inverse to solve, make sure that you realize that the answer you get comes from the range of the inverse function & you may need to use that as a reference angle(point) to find all possible values for original equation.

Recall:	$\sin(\sin^{-1} x)$	on $[-1, 1]$
	$\sin^{-1} (\sin x)$	on $[-\pi/2, \pi/2]$
	$\cos(\cos^{-1} x)$	on $[-1, 1]$
	$\cos^{-1}(\cos x)$	on $[0, \pi]$
	$\tan(\tan^{-1} x)$	on $(-\infty, \infty)$
	$\tan^{-1} (\tan x)$	on $(-\pi/2, \pi/2)$

5) Also realize that when finding all possible solutions that each of the functions repeats with a certain period and once the solutions are found in the fundamental period they can then be expanded to encompass all periods.

Recall: sine & cosine's period is 2π

$\therefore$ 2 values w/in fundamental cycle + $2k\pi$
where **k** is an integer

tangent's period is π

$\therefore$ 2 values w/in fundamental cycle + $k\pi$
where **k** is an integer

Example: Solve $3 \tan \theta - \sqrt{3} = 0$ on $[0, 360^\circ)$

Method 2: Factoring & Quadratic Methods

1) Get f(x) = 0 using algebra
2) Factor & use zero factor property
3) Use Method 1's process if needed

Example: Solve $\tan x \sin x + \sin x = 0$

Example: Solve $\cos \theta \cot \theta = -\cos \theta$ on $[0, 360^\circ)$

We just get more methods for solving and have the added problem that sometimes we will have **extraneous roots** that we will need to catch as we did when solving rational equations in algebra.

Method 3: Using Trig Identities

1) See a relationship that exists between 2 functions
2) Square sides in order to bring out an identity – be careful to check solutions for extraneous roots any time you do this
3) Use an identity to rewrite in terms of one function
4) Solve as in the last section

Example: Solve $\cot x - \sqrt{3} = \csc x$ on $[0, 2\pi)$

Note: *csc and cot are related via a Pythagorean Identity. You can create a problem by squaring to get there, so do check extraneous roots.*

Example: Solve $2\cos^2\theta - 2\sin^2\theta + 1 = 0$ on $[0, 360^\circ)$