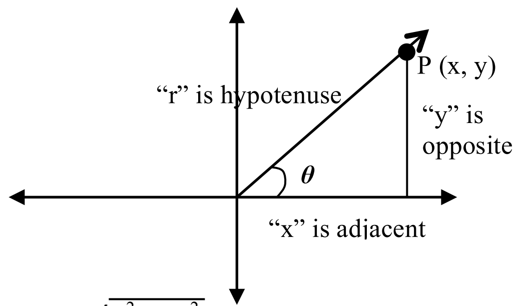
§8.6 Other Trigonometric Functions

I have already discussed all the trigonometric functions and their relationship to the sine and cosine functions and the x and y coordinates on the unit circle, but let's review.

Next we will define the trigonometric functions, review some basic geometry and make the connection to the special triangles that I asked you to memorize in the last section.

Standard Position of *t*



$r = \sqrt{x^2 + y^2}$ (Pythagorean Theorem)

$r > 0$ since it is a distance

(an <u>undirected vector</u> meaning it has no direction)

Based on the ***θ*** in Standard Position with a point on the terminal side defined as (x, y). The 6 trigonometric functions can be defined as seen below. The names of the 6 functions are sine, cosine, tangent, cotangent, secant and cosecant.

$\sin\theta = \dfrac{\text{opp}}{\text{hyp}} = \dfrac{y}{r}$ ***Note:*** When $r = 1$, $\sin\theta = y$

$\cos\theta = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{x}{r}$ ***Note:*** When $r = 1$, $\cos\theta = x$

$\tan\theta = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$ $\quad x \neq 0$

$\cot\theta = \dfrac{\text{adj}}{\text{opp}} = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta} = \dfrac{x}{y}$ $\quad y \neq 0$

$\sec\theta = \dfrac{\text{hyp}}{\text{adj}} = \dfrac{1}{\cos\theta} = \dfrac{r}{x}$ $\quad x \neq 0$

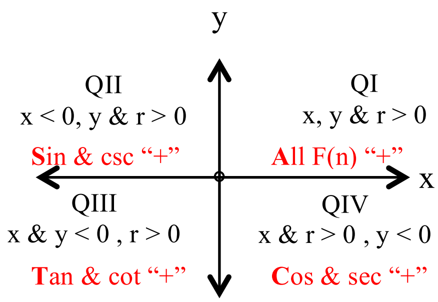
$\csc\theta = \dfrac{\text{hyp}}{\text{opp}} = \dfrac{1}{\sin\theta} = \dfrac{r}{y}$ $\quad y \neq 0$

Note: *These are the **<u>exact</u>** values for the 6 trig f(n). A calculator will yield only the approximate values of the functions.*

The exact values of all 6 trig functions can be found using the same principles that we applied earlier in finding the sine and cosine functions. Use reference angles in order to make your job easier. Remember the sign information "ASTC" to help use reference angles appropriately.

<u>Signs & Ranges of Function Values</u>

You don't have to memorize this, but you at least have to be able to develop it, which is dependent upon knowing quadrant information and standard position.



<u>This Saying Will Help Remember the Positive F(n)</u>

All	**A**ll f(n) "+"
Students	**s**in & csc "+"
Take	**t**an & cot "+"
Calculus	**c**os & sec "+"

Example: Fill in the following table using the definitions and identities of the 6 trig f(n).

Values of the 6 Trig F(n) for θ

θ	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0						
$\pi/2$						
π						
$3\pi/2$						
2π						

Example: Find the value of each using a reference angle

a) $\cot(-45^\circ)$ b) $\csc\left(-\frac{2\pi}{3}\right)$ c) $\sec\left(\frac{7\pi}{6}\right)$

Let's play around with "theoretical" relationships between the functions. We'll use the fundamental identities to rewrite each of the following into a single trig function.

Example: Rewrite as sin (θ), cos (θ), tan (θ) or cot (θ)

a) $\frac{1}{\csc(\theta)}$ b) $\frac{\sin(\theta)}{\tan(\theta)}$ c) $\frac{\sin(\theta)}{\sec(\theta)}$

Putting 2 concepts together we can solve the following problem. Concept one is the reciprocal identity and concept 2 is the know ratios and their relationships to the position on the unit circle.

Example: Find the 6 trig functions for θ if $\csc \theta = 2$ in QII
Precalculus with Limits, Hornsby, Lial, Rockswold, 5^{th} Ed. , #100 p. 592

At this point you should also know the domains & ranges of the 6 trig functions. The importance of this should be obvious once you have completed the above table and see which functions are undefined at what points. (That is showing you what is not in their domain.)

Domains of 6 Trig Functions

Sine and Cosine $\{ \boldsymbol{\theta} \mid \boldsymbol{\theta} \in \text{Real Number}\}$

Tangent and Secant $\{ \boldsymbol{\theta} \mid \boldsymbol{\theta} \neq {}^{(2n+1)\pi}/_2,\ n \in I\}$ (odd multiples of ${}^{\pi}/_2$)

Cotangent and Cosecant $\{ \boldsymbol{\theta} \mid \boldsymbol{\theta} \neq n\pi,\ n \in I\}$ (even multiples of ${}^{\pi}/_2$)

Range of 6 Trig Functions

Sine & Cosine $[-1, 1]$

Tangent & Cotangent $(-\infty, \infty)$

Secant & Cosecant $(-\infty, -1] \cup [1, \infty)$

Example: ID the quadrant(s) of an angle θ that satisfy the given conditions
Precalculus with Limits, Hornsby, Lial, Rockswold, 5^{th} Ed. , #68 & 76 p. 592

a) $\sin \theta > 0$ & $\tan \theta > 0$ b) $\cot \theta > 0$ & $\tan \theta > 0$

Example: Is the following statement possible or impossible for some θ?
Precalculus with Limits, Hornsby, Lial, Rockswold, 5^{th} Ed. , #86 & 90 p. 592

a) $\sec \theta = -0.9$ b) $\tan \theta = -6$ c) $\sin \theta = 1.01$

d) $\csc \theta = -5.01$

Next we must learn to graph the other 4 trig f(n). We will start with the tangent and its reciprocal identity the cotangent and then move to the reciprocal identities of sine and cosine.

The Domains:

Tangent	$\{x \mid x \neq {}^{(2n+1)\pi}/_2,\ n \in I\}$	(odd multiples of $^{\pi}/_2$)
Cotangent	$\{x \mid x \neq n\pi,\ n \in I\}$	(multiples of $^{\pi}/_2$)

Range:

Tangent & Cotangent $(-\infty, \infty)$

Period: The period, P, of $y = a \tan(x + \pi) = a \tan kx$ **and** $y = a \cot(x + \pi) = a \cot kx$ is given by

$$P = {}^{\pi}/_{k}$$

Amplitude: None exists

Vertical Asymptotes:

Because tan = $^{\sin}/_{\cos}$ so as cosine approaches zero (as it gets close to $^{\pi}/_2 \cdot (2n + 1)$)
tangent will take on infinitely large positive or negative values.

$\tan x \to -\infty$ as $x \to {}^{\pi}/_2{}^{+}$ (the little plus sign above and to the right means approaches from the right)

$\tan x \to \infty$ as $x \to {}^{\pi}/_2{}^{-}$ (the little minus above and to the right means approaches from the left)

Because cot = $^{\cos}/_{\sin}$ so as sine approaches zero (as it gets close to $n\pi$)
tangent will take on infinitely large positive or negative values.

$\cot x \to \infty$ as $x \to 0^{+}$ (the little plus sign above and to the right means approaches from the right)

$\cot x \to -\infty$ as $x \to \pi^{-}$ (the little minus above and to the right means approaches from the left)

Fundamental Cycle:

Tangent $(-^{\pi}/_2, {}^{\pi}/_2)$
Cotangent $(0, \pi)$

Even/Odd Characteristics:

Tangent is odd like its dominant parent the sine &
Cotangent is even like its dominant parent the cosine

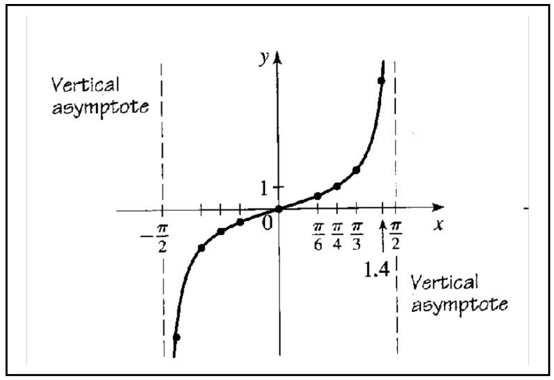
Note: Symmetric about the origin
When $\tan(-x) = -\tan(x)$

Note: Symmetric about the y-axis
When $\cot(-x) = \cot(x)$

Tables & Graphs:

x	y = tan x
$-\pi/2$	undefined
$-\pi/4$	-1
0	0
$\pi/4$	1
$\pi/2$	undefined

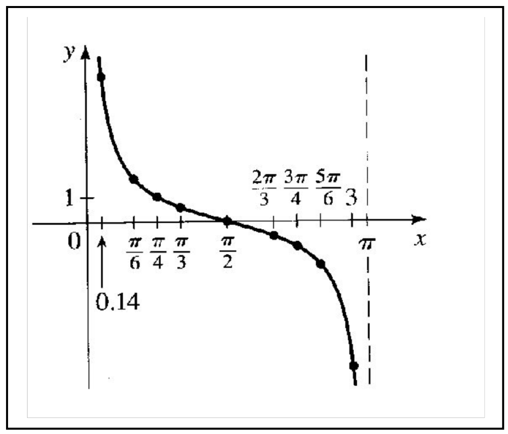
* For y = a tan kx use $(-\pi/_{2k}, \pi/_{2k})$



Note: *Scan from p. 435 of Stewart*

x	y = cot x
0	undefined
$\pi/4$	1
$\pi/2$	0
$3\pi/4$	-1
π	undefined

* For y = a cot kx use $(0, \pi/_k)$



Note: *Scan from p. 435 of Stewart*

Tangent and Cotangent functions can have period changes, phase shifts, reflections and vertical translation. Although stretching does occur it is not a change in amplitude.

y = A tan (k[x – B]) + C **or** y = A cot (k[x – B]) + C

A changes 1 & -1 ➜ A•1 & A• -1 (the y-values)

k changes the period π ➜ $\pi/_k$

B changes the values of x used

$(-\pi/2 + B), (-\pi/4 + B), (0 + B), (\pi/4 + B), (\pi/2 + B)$ **or** $(0 + B), (\pi/4 + B), (\pi/2 + B), (3\pi/4 + B), (\pi + B)$

C changes 0, 1 & -1 or 0, A•1 & A•-1 ➜

0 + C, 1 + C or 1•A + C & -1 + C or -1•A + C

Note: *Asymptotic shifts also occur based on B*

$(-\pi/2 + B)$ & $(\pi/2 + B)$ for tangent **and** $(0 + B)$ & $(\pi + B)$ for cotangent

Example: Give a table showing the translation & graph

$$y = 2 \tan x$$

Example: Give a table showing the translation & graph

$$y = \cot {}^{1}/_{2}x$$

Example: Give a table showing the translation & graph

$$y = -\tan (x + {}^{\pi}/_{2})$$

Finally, we will discuss the graphs for the reciprocal identities for sine and cosine.

The Domains:

Cosecant	$\{x \mid x \neq n\pi,\ n \in I\}$	(multiples of $^{\pi}/_{2}$)
Secant	$\{x \mid x \neq {}^{(2n+1)\pi}/_{2},\ n \in I\}$	(odd multiples of $^{\pi}/_{2}$)

Range:

Cosecant & Secant $(-\infty, -1] \cup [1, \infty)$

Period: The period, P, of $y = a \csc(x + 2\pi) = a \csc kx$ **and** $y = a \sec(x + 2\pi) = a \csc kx$

is given by

$$P = {}^{2\pi}/_{k}$$

Amplitude: None exists

Vertical Asymptotes:

Because $\csc = {}^{1}/_{\sin}$ so as sine approaches zero (as it gets close to $n\pi$; mult. of π) cosecant will take on infinitely large positive or negative values.

$\csc x \to \infty$ as $x \to 0^{+}$

$\csc x \to \infty$ as $x \to \pi^{-}$

$\csc x \to -\infty$ as $x \to \pi^{+}$

$\csc x \to -\infty$ as $x \to 2\pi^{-}$

Because $\sec = {}^{1}/_{\cos}$ so as cosine approaches zero (as it gets close to $^{\pi}/_{2}(2n + 1)$; odd mult. of $^{\pi}/_{2}$) secant will take on infinitely large positive or negative values.

$\sec x \to \infty$ as $x \to {}^{3\pi}/_{2}{}^{+}$

$\sec x \to \infty$ as $x \to {}^{\pi}/_{2}{}^{-}$

$\sec x \to -\infty$ as $x \to {}^{\pi}/_{2}{}^{+}$

$\sec x \to -\infty$ as $x \to {}^{3\pi}/_{2}{}^{-}$

Even/Odd Characteristics:

Cosecant is odd like its parent the sine & Secant is even like its parent the cosine

Note: Symmetric about the origin
When $\csc(-x) = -\csc(x)$

Note: Symmetric about the y-axis
When $\sec(-x) = \sec(x)$

Fundamental Cycle/Primary Interval:

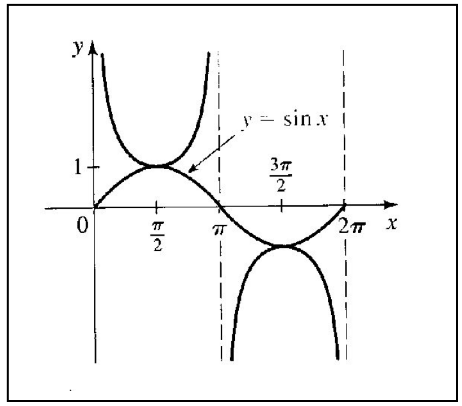
Cosecant $(0, 2\pi)$

Secant $(0, 2\pi)$

Tables & Graphs:

x	y = csc x
0	undefined
$^{\pi}/_{2}$	1
π	undefined
$^{3\pi}/_{2}$	-1
2π	undefined

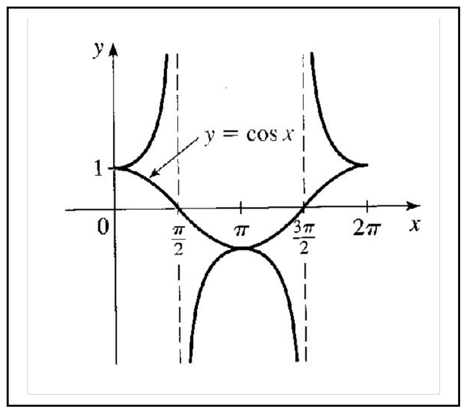
* For y = a csc *k*x use $(0, {}^{2\pi}/_{k})$



***Note:** Scan from p. 435 of Stewart*

x	y = sec x
0	1
$^{\pi}/_{2}$	undefined
π	-1
$^{3\pi}/_{2}$	undefined
2π	1

* For y = a sec *k*x use $(0, {}^{2\pi}/_{k})$



***Note:** Scan from p. 435 of Stewart*

Cosecant and Secant functions can have period changes, phase shifts, reflections and vertical translation. Although stretching does occur it is not a change in amplitude.

$y = A \csc (k[x - B]) + C$ **or** $y = A \sec (k[x - B]) + C$

A changes 1 & -1 ➔ A•1 & A• -1 (the y-values)

k changes the period 2π ➔ $^{2\pi}/_{k}$

B changes the values of x used

$(0 + B), ({}^{\pi}/_{2} + B), (\pi + B), ({}^{3\pi}/_{2} + B), (2\pi + B)$

C changes 1 & -1 or A•1 & A•-1 ➔ 1 + C or 1•A + C & -1 + C or -1•A + C

***Note:** Asymptotic shifts also occur based on B*

$(0 + B)$ & $(\pi + B)$ & $(2\pi + B)$for cosecant **and** $({}^{\pi}/_{2} + B)$ & $({}^{3\pi}/_{2} + B)$ for secant

Example: Give a table showing the translation & graph

$y = 2 \csc x$

Example: Give a table showing the translation & graph

$y = \sec {}^{1}/_{2}x$

Example: Give a table showing the translation & graph

$y = -\csc (x + {}^{\pi}/_{2})$

§8.7 The Inverse Circular F(n)

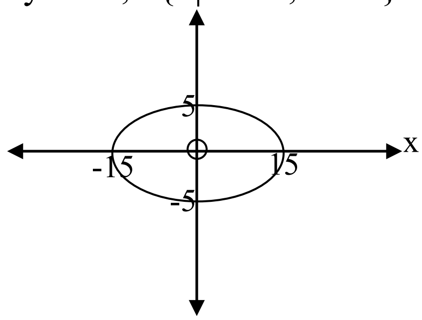
First let's review relations, functions and one-to-one functions.

A **relation** is any set of ordered pairs. A relation can be shown as a set, a graph or a function (equation).

a) $\{(2,5), (2,6), (2,7)\}$

b) $y = \sqrt{x}, \quad \{x \mid x \geq 0, x \in \Re\}$

c)



A **function** is a relation which satisfies the condition that for each of its independent values (x-values; domain values) there is only 1 dependent value (y-value; f(x) value; range value).

From above, only satisfies this requirement b) $y = \sqrt{x}, \quad \{x \mid x \geq 0, x \in \Re\}$

Note: *We can check to see if a relation is a f(n) by seeing if any of the x's are repeated & go to different y's; on a graph a relation must pass vertical line test and if it's a f(n)/equation then you can think about it in terms of the domain and range or in terms of its graph.*

A **one-to-one function** is a function that satisfies the condition that each element in its range is used only once (has a unique x-value; domain value).

From above, only satisfies this requirement b) $y = \sqrt{x}, \quad \{x \mid x \geq 0, x \in \Re\}$

Note: *We can check to see if a function is 1:1 by seeing if any of the y's are repeated & go to different x's; on a graph a 1:1 f(n) must pass horizontal line test and if it's a f(n)/equation then you can think about it in terms of the domain and range or in terms of its graph.*

If you remember from your study of Algebra, we care about one-to-one functions because they have an **inverse.** The **inverse** of a function, written $f^{-1}(x)$, is the function for which the domain and range of the original function f(x) have been reversed. The composite of an inverse and the original function is always equal to x.

Inverse of f(x)

$f^{-1}(x) = \{(y, x) \mid (x, y) \in f(x)\}$

__Note:__ $f^{-1}(x)$ is not the reciprocal of f(x) but the notation used for an inverse function!! The same will be true with our trigonometric functions.

<u>Facts About Inverse Functions</u>

1) Function must be 1:1 for an inverse to exist; we will sometimes restrict the domain of the original function, so this is true, but only if the range is not effected. ***Note:*** *You will see this with the trig functions because they are not 1:1 without the restriction on the domain.*
2) (x, y) for f(x) is (y, x) for $f^{-1}(x)$
3) f(x) and $f^{-1}(x)$ are reflections across the y = x line
4) The composite of f(x) with $f^{-1}(x)$ or $f^{-1}(x)$ with f(x) is the same; it is x
5) The inverse of a function can be found by changing the x and y and solving for y and then replacing y with $f^{-1}(x)$. ***Note:*** *This isn't that important for our needs here.*

What you should take away from the above list is:

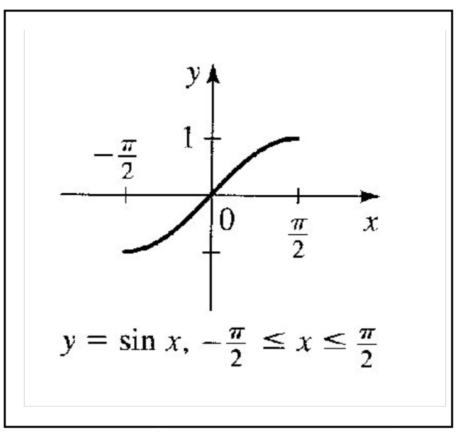
1) We make the trig functions one-to-one by restricting their domains
2) (x, y) is (y, x) for the inverse function means that if we have the value of the trig function, the y, the inverse will find the angle that gives that value, the x
3) When we graph the inverse functions we will see their relationship to the graphs of the corresponding trig function as a reflection across y = x.
4) $\sin^{-1}(\sin x)$ is x
   Note: *You might use this one in your Calculus class.*

<u>The Inverse Sine – Also called the ArcSin</u>

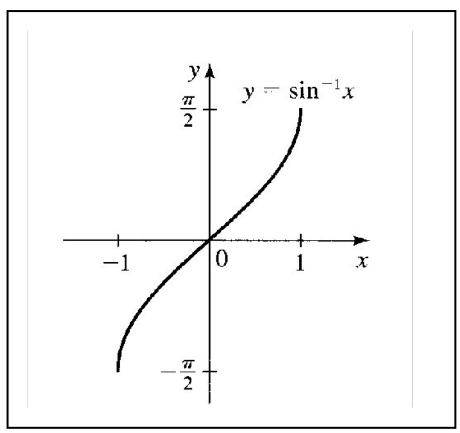
If f(x) = sin x on D: $[-\pi/2, \pi/2]$ **&** R: [-1, 1]

Notice: Restriction to the original domain is to QI & QIV of unit circle!

then $f^{-1}(x) = \sin^{-1} x$ **or** $f^{-1}(x) = \arcsin x$ on D: [-1, 1] **&** R: $[-\pi/2, \pi/2]$



Note: *Scan from p. 551 Stewart ed 5*



Note: *Scan from p. 551 Stewart ed 5*

<u>Finding y = arcsin x</u>

1) Think of arcsin as finding the x-value (the radian measure or degree measure of the angle) of sin that will give the value of the argument.

   Rewrite y = arcsin x to sin y = x where y = ? if it helps.

2) Think of your triangles! What ∠ are you seeing the opposite over hypotenuse for?

Example: Find the exact value for **y** in terms of 0 to $^{\pi}/_{2}$ without a calculator. Don't forget to check domains! Use radian measure to give the angle.

a) $y = \arcsin {}^{\sqrt{3}}/_{2}$ b) $y = \sin^{-1}(^{-1}/_{2})$ c) $y = \sin^{-1}\sqrt{2}$

What other angle(s) are in the interval [0, 2π] are possible solutions to a) & b)?

However, it is not always possible to use our prior experience with special triangles to find the inverse. Sometimes we will need to use the calculator to find the inverse.

Example: Find the value of the following in radians rounded to 5 decimals

*Precalculus, 6th ed., Stewart, Redlin & Watson p. 467 #10

*a) $y = \sin^{-1}(^{1}/_{3})$ b) $y = \sin^{-1}(-^{1}/_{3})$

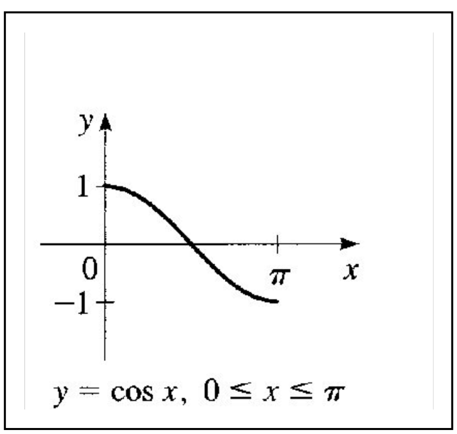
What other angle(s) are in the interval [0, 2π] are possible solutions to a) & b)?

<u>The Inverse Cosine – Also called the ArcCos</u>

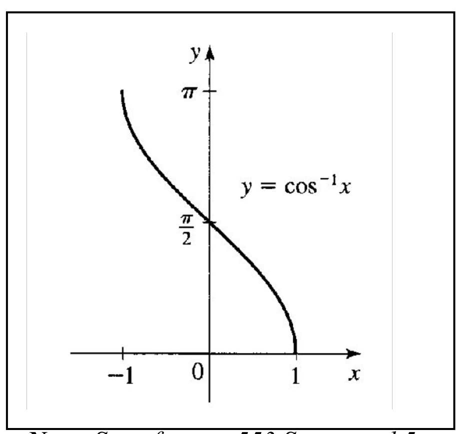
If $f(x) = \cos x$ on D: [0, π] **&** R: [-1, 1]

Notice: Restriction to the original domain is to QI & QII of unit circle!

then $f^{-1}(x) = \cos^{-1} x$ **or** $f^{-1}(x) = \arccos x$ on D: [-1, 1] **&** R: [0, π]



Note: *Scan from p. 553 Stewart ed 5*



Note: *Scan from p. 553 Stewart ed 5*

Example: Find the exact value of **y** for each of the following in radians.

a) $y = \arccos 0$ b) $y = \cos^{-1}(^1/_2)$

Example: Find the value of the following in radians rounded to 5 decimals.

*Precalculus, 6th ed., Stewart, Redlin & Watson p. 467 #8

$$y = \cos^{-1}(-0.75)$$

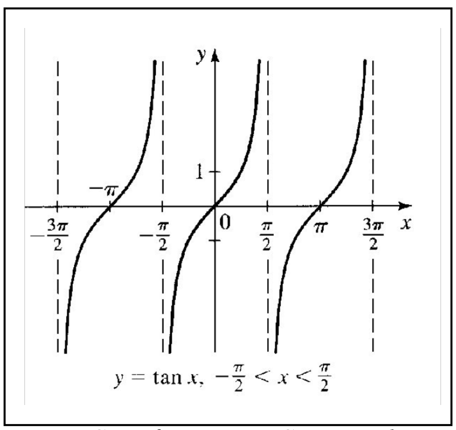
What other value(s) in the interval [0, 2π] does this represent?

__**The Inverse Tangent – Also called the ArcTan**__

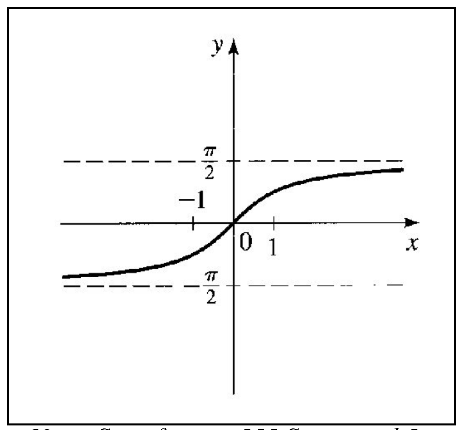
If $f(x) = \tan x$ on D: $[^{-\pi}/_2, {}^{\pi}/_2]$ **&** R: $[-\infty, \infty]$

Notice: Restriction to the original domain is to QI & QIV of unit circle!

then $f^{-1}(x) = \tan^{-1} x$ **or** $f^{-1}(x) = \arctan x$ on D: $[-\infty, \infty]$ **&** R: $[^{-\pi}/_2, {}^{\pi}/_2]$



Note: *Scan from p. 555 Stewart ed 5*



Note: *Scan from p. 555 Stewart ed 5*

Notice: *The inverse tangent has horizontal asymptotes at* $\pm^{\pi}/_2$. *It might be interesting to note that the inverse tangent is also an odd function (recall that tan (-x) = -tan x and also the arctan (-x) = - arctan (x)) just as the tangent is and that both the x & y intercepts are zero as well as both functions being increasing functions.*

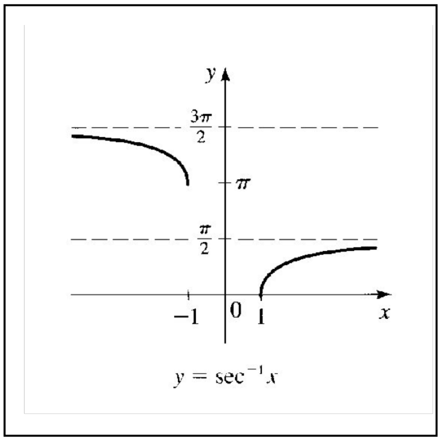
Example: Find the exact value for **y,** in radians without a calculator.

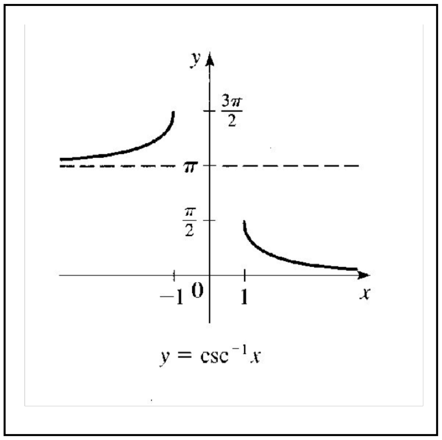
$$y = \arctan \sqrt{3}$$

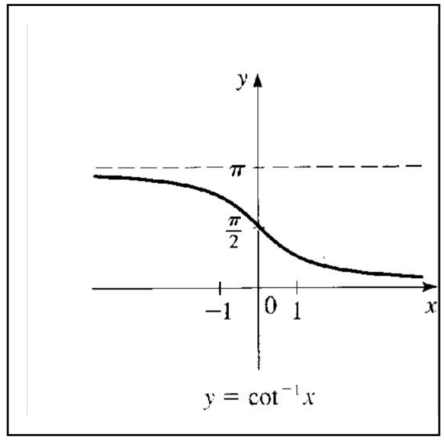
I am not going to spend a great deal of time talking about the other 3 inverse trig functions in class or quizzing/testing your graphing skills for these trig functions. I do expect you to know their domains and ranges and how to find exact and approximate values for these functions.

Summary of Sec^{-1}, Csc^{-1} and Cot^{-1}

Inverse Function	Domain	Interval	Quadrant on Unit Circle
$y = \cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$	I & II
$y = \sec^{-1} x$	$(-\infty,-1] \cup [1,\infty)$	$[0, \pi]$, $y \neq \pi/2$	I & II
$y = \csc^{-1} x$	$(-\infty,-1] \cup [1,\infty)$	$[-\pi/2, \pi/2]$, $y \neq 0$	I & IV







Example: Find the exact value for **y,** in radians without a calculator.

$$y = \text{arccsc } -\sqrt{2}$$

Before we do any examples that require the use of our calculator to find the inverse of these functions, it is important that you recall your reciprocal identities! It is by using these reciprocal identities first that you can find the inverse values on your calculator. Let me show you an example first.

Example: Find the approximate value in radians rounded to the nearest 5 decimals.

$$y = \sec^{-1}(-4)$$

STEP 1: Rewrite as secant $\sec y = -4$

STEP 2: Rewrite both sides by taking the reciprocal $\frac{1}{\sec y} = \frac{-1}{4}$

STEP 3: Use reciprocal identity to rewrite $\cos y = \frac{-1}{4}$

STEP 4: Find the inverse of both sides to solve for y

$$\cos^{-1}(\cos y) = \cos^{-1}\left(\frac{-1}{4}\right)$$

STEP 5: Use your calculator to find $y = \cos^{-1}\left(\frac{-1}{4}\right)$

Example: Find the approximate value in radians rounded to the nearest 5 decimals.

$$y = \text{arccot}(-0.2528)$$

Now, we will expand the range of our functions by messing with the domain values. For more information see p. 597 Example #7 of your book. First, adjust the domain and find the values within this domain that satisfy the requirements and then solve for the angle. Note that this is a substitution problem cos (A) = y, but A is actually a different value that involves the angle so you must solve for the angle.

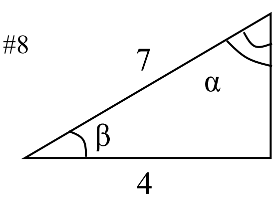
Example: Let's do #15 & #20 from p. 605 of your text. Find all values on $[0, 2\pi]$ in a) and $[0, 360^\circ]$ on b).

**Precalculus: A Make it Real Approach, Wilson, Adamson, Cox & O'Bryan*

a) $\cos\left(\frac{1}{2}\theta\right) = -\frac{\sqrt{3}}{2}$ b) $1.5\tan(\theta + 45^\circ) = 3$

Example: Find the 6 trig functions for α and β, and then find the value of α and β to correct to 5 decimals.

*Precalculus, 5th ed., Stewart, Redlin & Watson p. 484 #8



We can also "take a page" from the skill-set that we have just been using and use it to solve a little more difficult problems – those that deal with the composite of trig and inverse trig functions. In these type of problems we will work from the inside out. We **don't actually ever get a degree measure** from these problems, because the final answer is actually the ratio of sides.

1) We will use the inside function to put together a triangle
2) Then we use the Pythagorean Theorem to find the missing side
3) Finally, we use the definitions of the trig functions based upon opposite, adjacent and hypotenuse lengths to answer the problem.

Example: Evaluate each of the following without a calculator

a) $\cos\left(\sin^{-1}\frac{2}{3}\right)$ b) $\sec\left(\cot^{-1}\left(\frac{-15}{8}\right)\right)$

Let's do an application problem that your book doesn't do, but are important application problem.

Example: Find the angle of elevation of the sun.

***Precalculus*, 6th ed., Stewart, Redlin & Watson p. 468 #38