

## §8.4 Graphs of Sine and Cosine Functions

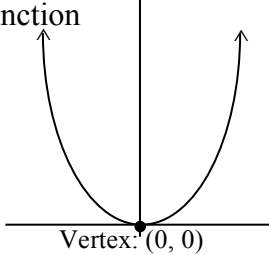
### Additional Material to Assist in Graphing Trig Functions

One of the things that will help a great deal in learning to graph the trig functions is an understanding of translation. I'm going to go over the translation of a quadratic function to assist you in learning how to graph trig functions.

- 1) Every function has a general form of the equation and a graph centered at the origin.
- 2) It is from this general form that translations happen. Think of a translation as moving the shape formed from the general equation around in space. We can move the shape up, down, left, right, flip it over or stretch/shrink it. It really gets fun when we do multiple movements!
  - a) Stretching/Shrinking → Multiplies the function value (the y-value) by a constant
  - b) Reflection (Flipping it over) → Multiplies the function by a negative
  - c) Vertical Translation (Moving it up/down) → Adds a constant to the function value (the y-value)
  - d) Horizontal Translation (Moving it left/right) → Adds a constant value to the x-value while still outputting the same y-value

Let's go through this with the quadratic function:  $y = x^2$

- 1) First you must know the basics about the general function
  - a) This function forms a parabola with its vertex at (0, 0)
 


  - b) The general parabola (being an even function) is symmetric about the y-axis; the line to which the parabola is symmetric is called the line of symmetry
  - c) The vertical line through the vertex is called the line of symmetry
    - i) Relating to being an even function
    - ii) For every  $f(-x)$  there is an equivalent  $f(x)$

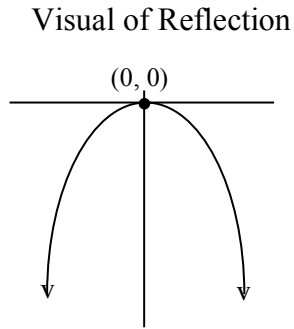
Now, let's look at a typical table of values that we use to graph this function. This will assist in seeing the translations of this function.

x	$y = x^2$
0	0
1	1
-1	1
2	4
-2	4
3	9
-3	9

Let's take our first translation to be the **reflection**. This simply multiplies the y-value by a negative.

↓ reflection

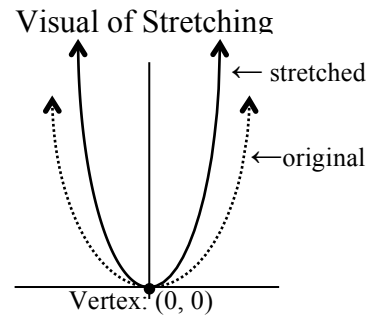
x	$y = x^2$	$y = -x^2$
0	0	0
1	1	-1
-1	1	-1
2	4	-4
-2	4	-4
3	9	-9
-3	9	-9



Next, take a **stretching/shrinking** translation. This multiplies the y-value by a negative. Visually it is like “pulling the parabola up by its ends or pushing it down.” When the constant is  $> 1$  the parabola is stretched and when it is  $< 1$  but  $> 0$  it is shrunk.

↓ stretching

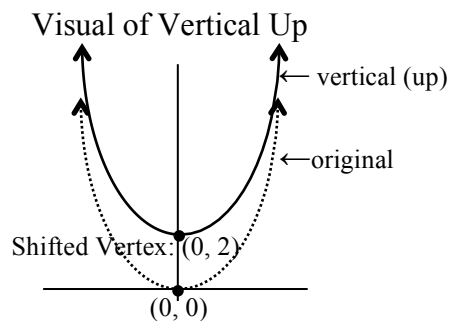
x	$y = x^2$	$y = 3x^2$
0	0	0
1	1	3
-1	1	3
2	4	12
-2	4	12
3	9	27
-3	9	27



Next, take the **vertical** translation. This adds to the y-value. Visually it moves the parabola up and down the y-axis. When a constant is added to the function, the translation is up and when the constant is subtracted from the translation is down.

↓ vertical

x	$y = x^2$	$y = x^2 + 2$
0	0	2
1	1	3
-1	1	3
2	4	6
-2	4	6
3	9	11
-3	9	11



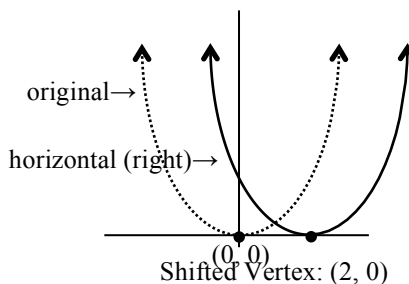
Last, the most difficult translation to deal with in terms of ordered pairs, because it changes the  $x$  not the  $y$ -coordinate. This is the **horizontal** translation. Visually it moves the parabola to the left or right. When a *constant is subtracted* from the  $x$ -value, the translation is *right* and when it is added it is to the left (this is the opposite of what you think, and it is due to the form that the equations take).

↓ horizontal

$x$	$x'$	$y = (x - 2)^2$
0	2	0
1	3	1
-1	1	1
2	4	4
-2	0	4
3	5	9
-3	-1	9

↑ there's notice no change from  $y = x^2$

Visual of Horizontal Right



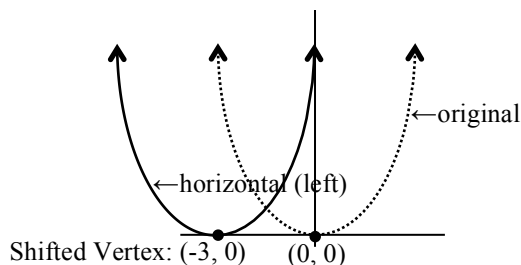
Because this one is the mind bender, I will also include the left shift.

↓ horizontal

$x$	$x'$	$y = (x + 3)^2$
0	-3	0
1	-2	1
-1	-4	1
2	-1	4
-2	-5	4
3	0	9
-3	-6	9

↑ there's notice no change from  $y = x^2$

Visual of Horizontal Left



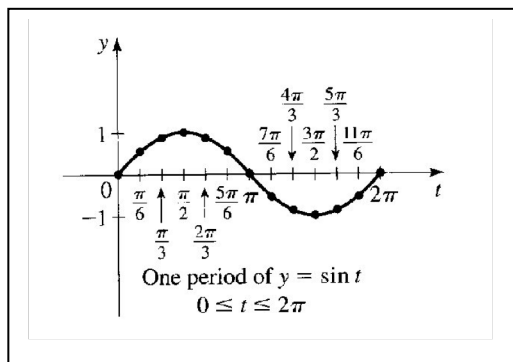
Now we can return to the task at hand and apply this same knowledge to trig functions. To do this we will:

- 1) Learn the basic shapes of the trig function graphs
- 2) Learn the special names that go along with the translations
- 3) See each of the translations apply to the trig function graphs
  - a) Recognize the translations from an equation (and eventually the opposite; make equations from translation recognition)
  - b) Focus on tabular values for the translations
  - c) Take tabular values to graph the function

## Trig Graphs

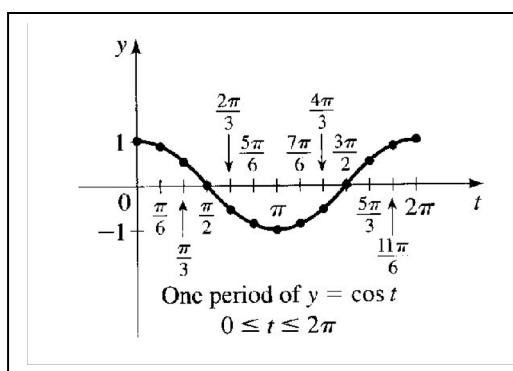
First let's look at the two most basic trig function graphs. These are the graphs of the sine and the cosine. We call the visual representations of the sine and cosine functions **sinusoids**. We will be graphing the **fundamental cycle** of the trig functions. The fundamental cycle for the sine and cosine function is on the interval  $[0, 2\pi]$ . After we look at tables and graphs for each of the functions we will get into translations and the special definitions that go along with the translations.

$t$	$y = \sin t$
0	0
$\frac{\pi}{2}$	1
$\pi$	0
$\frac{3\pi}{2}$	-1
$2\pi$	0



*Note: Scan from p. 420 Stewart*

$t$	$y = \cos t$
0	1
$\frac{\pi}{2}$	0
$\pi$	-1
$\frac{3\pi}{2}$	0
$2\pi$	1



*Note: Scan from p. 420 Stewart*

Notice that I have not included all of the intermediate values in my tables. If you know the general shape of the curve, the basic shape can be drawn based on the minimum and maximum values and a point midway between. As you can see, the function values for sine and cosine are the same values that we see on the unit circle as the y-coordinate for the sine function and as the x-coordinate for the cosine function. The domain of the functions are the values that  $t$  can take on. We graph one period of each function where  $0 \leq t \leq 2\pi$  unless otherwise stated.

Next we will investigate the moving of these visual representations of the functions around in space, just as we saw happening with the quadratic function.

If  $y = f(x)$  is a function and “ $a$ ” is a nonzero constant such that  $f(x) = f(x + a)$  for every “ $x$ ” in the domain of  $f$ , then  $f$  is called a **periodic function**. The smallest positive constant “ $a$ ” is called the **period** of the function,  $f$ .

**Period Theorem:** The period,  $P$ , of  $y = \sin(Bx)$  and  $y = \cos(Bx)$  is given by

$$P = \frac{2\pi}{B}$$

The **amplitude** of a sinusoid is the absolute value of half the difference between the maximum and minimum y-coordinates.

**Amplitude Theorem:** The amplitude of  $y = A \sin x$  or  $y = A \cos x$  is  $|A|$ .

*Note: The amplitude is the stretching/shrinking translation*

The **phase shift** of the graph of  $y = \sin(x - C)$  or  $y = \cos(x - C)$  is  $C$ .

*Note: The phase shift is the horizontal translation*

### General Sinusoid:

The graph of:  $y = A \sin(k[x - B]) + C$  *or*  $y = A \cos(k[x - B]) + C$

Is a sinusoid with:      Amplitude =  $|A|$   
                                    Period =  $\frac{2\pi}{k}$ , ( $k > 0$ )  
                                    Phase Shift =  $B$   
                                    Vertical Translation =  $C$

### Process For Graphing a Sinusoid Based on Translation

1. Sketch one cycle of  $y = \sin kx$  or  $y = \cos kx$  on  $[0, \frac{2\pi}{k}]$ 
  - a) Change  $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$  into the appropriate values based on  $\frac{2\pi}{k}$  and 4 evenly spaced values.
2. Change the amplitude of the cycle according to the value of  $A$ 
  - a) Take all general function values of  $y$  and multiply them by  $A$
3. If  $A < 0$ , reflect the curve about the x-axis
  - a) Take all values of  $y$  from 2a) and make them negative
4. Translate the cycle  $|B|$  units to the right if  $B > 0$  and to the left if  $B < 0$ 
  - a) Add  $B$  to the x-values in 1a)
5. Translate the cycle  $|C|$  units upward if  $C > 0$  or downward if  $C < 0$ 
  - a) Add  $C$  to the y-values in 3a)

**Example:** Graph  $y = 2 \sin (3[x + \pi/3]) + 1$

1. **Find the period & compute the 4 evenly spaced points used to graph**  
 $y = \sin 3x$  on  $[0, 2\pi/3]$ 
  - a) The x-values are now  $0, (2\pi/3 \cdot 1/4), (2\pi/3 \cdot 1/2), (2\pi/3 \cdot 3/4), 2\pi/3$   
or  $[0, \pi/6, \pi/3, \pi/2, 2\pi/3]$  *Note: The multiplication by  $1/4, 1/2$  &  $3/4$  yields 4 points evenly space over one period.*
2. **Find the amplitude & translate the y-values by multiplication**  
 $y = 2 \sin 3x$ 
  - a) Sine y-values are usually 0, 1, 0, -1, 0 so now they are 0, 2, 0, -2, 0
3. **Investigate any possible reflection**  
A is positive so no change occurs here.
  - a) Sine y-values remain as in 2a)
4. **Compute the phase shift (horizontal translation) & translate x-values**  
*Note: The phase shift can't be read until it is written as  $k[x - B]$  – this may require a little algebraic manipulation*  
 $y = 2 \sin (3[x - (-\pi/3)])$ 
  - a) Add B to x-values in 1a) They were  $0, \pi/6, \pi/3, \pi/2, 2\pi/3$ , so  
now they are  $(0 - \pi/3), (\pi/6 - \pi/3), (\pi/2 - \pi/3), (2\pi/3 - \pi/3)$   
or  $[-\pi/3, -\pi/6, 0, \pi/6, \pi/3]$
5. **Translate vertically by adding to the last translation of the y-values from either 2a) or 3a)**  
 $y = 2 \sin (3[x + \pi/3]) + 1$ 
  - a) The y-values from 2a) or 3a) have 1 added to them.  
They were 0, 2, 0, -2, 0 so they are now 1, 3, 1, -1, 1

This is a table that shows the translations. You would plot the points from the furthest right x & furthest right y.

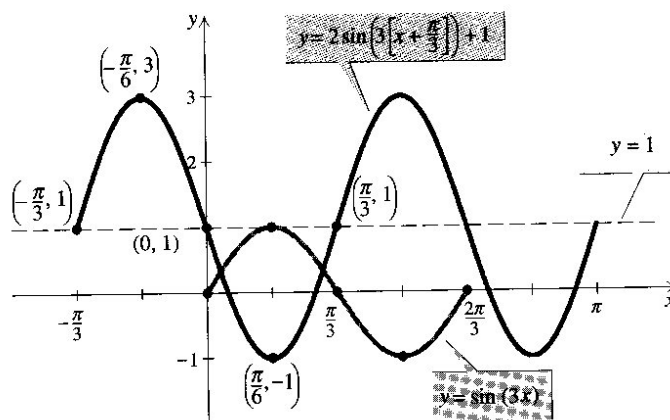
Plot

Plot



t for sin t	t' for sin 3t	t'' for $\sin(3[t - (-\pi/3)])$	Y for sin t	Y for 2 sin 3t	Y for f(t)
0	0	$-\pi/3$	0	0	1
$\pi/2$	$\pi/6$	$-\pi/6$	1	2	3
$\pi$	$\pi/3$	0	0	0	1
$3\pi/2$	$\pi/2$	$\pi/6$	-1	-2	-1
$2\pi$	$2\pi/3$	$\pi/3$	0	0	1

And here is your graph with the translation from  $y = \sin 3x$  to  $f(x)$



*Note: This example is from Precalculus Functions & Graphs and Precalculus with Limits, Mark Dugopolski*

- Example:**
- What translation is being done to  $y = 3 + \sin x$ ?
  - Give a table using the same values of x for  $y = \sin x$  &  $y = 3 + \sin x$
  - Sketch both sinusoids on the same graph  
(Similar #2 p. 429 Stewart)

- Example:**
- a) What translation(s) are being done to  $y = \cos 3x$ ?
  - b) What is the amplitude of the function?
  - c) What is the period of the function?
  - d) Give a table using the same values of  $x$  for  $y = \cos x$  &  $y = \cos 3x$
  - e) Sketch both sinusoids on the same graph

- Example:**
- a) What translation(s) are being done to  $y = 5 \sin^{1/4}x$ ?
  - b) What is the amplitude of the function?
  - c) What is the period of the function?
  - d) Give a table using the same values of  $x$  for  $y = \cos x$  &  $y = 5 \sin^{1/4}x$
  - e) Sketch both sinusoids on the same graph

- Example:**
- a) What translation(s) are being done to  $y = -4 \cos^{1/2}(x + \pi/2)$ ?
  - b) What is the amplitude of the function?
  - c) What is the period of the function?
  - d) What is the phase shift?
  - e) Give a table using the same values of  $x$  for  $y = \cos x$  &  $y = -4 \cos^{1/2}(x + \pi/2)$
  - f) Sketch both sinusoids on the same graph



**Example:** You can't see the period or the phase shift as this function is currently written. Re-write the function and give the **period** and the **phase shift**. (Adapted from #39 p. 429 Stewart)

$$y = \sin(\pi + 3x)$$

### Using a Graphing Calculator to View a Sinusoid

- 1) Find the period, amplitude and phase shift and use these to set the viewing window (See p. 426 of Stewart's text discusses the dangers of a viewing window that is not correctly set. The summary is you need a viewing window that reflects the period and amplitude.)
- 2) Put your calculator in RADIAN mode
- 3) Use the  $\boxed{Y=}$  key to enter the trig function; use  $\boxed{X,T, \theta, n}$  key for the t value
- 4) ZOOM trig to graph
- 5) Use WINDOW to set the view rectangle appropriately

**Example:** Let's try this together and at the same time demonstrate 1) translations and 2) inappropriate viewing rectangle.

- a)  $\sin 50x$
- b)  $-1 + \sin 50x$
- c)  $0.5 \sin 50x$
- \*d)  $\sin 50x - 1$

\*This is the example that could create havoc if you try to create the wrong viewing window to see all of these at the same time.

### Application of Sinusoids

Frequency is a real world interpretation of the period of a sinusoid. The frequency is the reciprocal of the period. It tells us how many cycles are in one interval of time. A cycle is what we see as the defining characteristics of a period – the highs, lows and middle. This is interpreted in physics for light, sound and energy. You will see applications ranging from music to electrical engineering and astronomy.

$$F = 1/p \quad \text{or} \quad F = k/2\pi \quad \text{for} \quad y = A \sin(k[x - B]) + C \quad \text{or} \quad y = A \cos(k[x - B]) + C$$

**Example:** Give the frequency of the  $y = \cos(0.001\pi x)$  where x is in seconds

\* *Trigonometry*, Dugopolski, 3<sup>rd</sup> Edition, #42 p.127

## Curve Fitting for Sinusoids

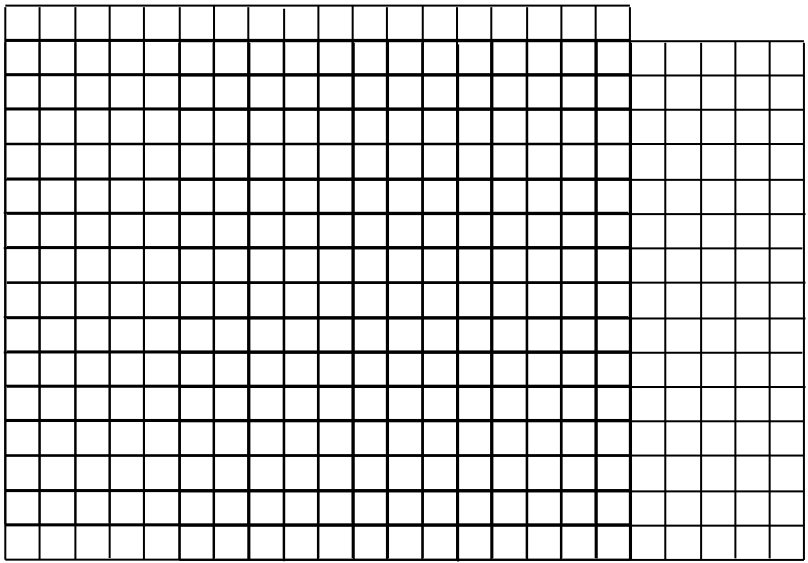
### Steps to Take

1. Plot the points given on a coordinate system.
2. Decide if your graph appears to look more like a sine or cosine function and start with the appropriate sine function in the form:  
$$y = A \sin (k[x - B]) + C$$
3. Find the amplitude A  
$$A = \frac{1}{2} (\text{maximum } y - \text{minimum } y)$$
4. Find the vertical shift B  
$$B = \frac{1}{2} (\text{maximum } y + \text{minimum } y)$$
5. Find the period k  
$$k = 2\pi \div [2(\text{x of maximum } y - \text{x of minimum } y)]$$
6. Find the phase shift C  
$$C = \text{x of the maximum } y$$
7. Put the A, B, k & C appropriately into the sine function in #1

**Example:** Use a sine function to model the following. We are using a sine because a plot of the points reveals a curve that looks like a sine wave.\* *Precalculus: Mathematics for Calculus*, Stewart, 5<sup>th</sup> ed., p. 464

*Circadian rhythm is the daily biological pattern by which body temperature, blood pressure, and other physiological variables change. The data in the table below show typical changes in human body temperature over a 24-hour period where  $t = 0$  corresponds to midnight.*

Time After Midnight	Body Temperature °C
0	36.8
2	36.7
4	36.6
6	36.7
8	36.8
10	37.0
14	37.3
16	37.4
18	37.3
20	37.2
22	37.0
24	36.8



### **Modeling Sinusoids Using Your Calculator**

1. Input data into the STAT Editor into 2 registers  
     STAT → ENTER → Input Independent Values in L<sub>1</sub>  
     Use right arrow key to move to L<sub>2</sub> and Input Dependent Variables
2. Make sure that you are using radian mode
3. Find the SinReg function. You might find it in STAT→TESTS or you might need to use the CATALOG of functions.  
     2<sup>nd</sup> 0 then ALPHA LN then use arrow down to find SinReg in CATALOG
4. Give the registers for the independent and dependent variables  
     2<sup>nd</sup> 1 if you used L<sub>1</sub> for indep , 2<sup>nd</sup> 2 if you used L<sub>2</sub> for dependent
5. Read the values for the sine and write appropriately.
6. If comparing to a hand derived cosine function remember to subtract  $\pi/2$  (approx. 1.57) from the calculator's "c" value and then factor so you can see the same function that you arrived at by hand.
7. If comparing to a hand derived sine function don't forget to factor.

\*Here is [a link](http://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CDkQFjAB&url=http%3A%2F%2Fwww.austincc.edu%2Fjthom%2FSineRegressionOnTIs.pdf&ei=6vnSUtD7MabQsATKzIGADA&usg=AFQjCNF-sJS1KCsu2KCzIYldLGyPhBob_g&sig2=Vv9gKOJQoVF0ix1eNCE8NA&bvm=bv.59026428,d.cWc) to someone else's calculator handout that I found on the internet. (Here is the link so you can type it in as well:

[http://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CDkQFjAB&url=http%3A%2F%2Fwww.austincc.edu%2Fjthom%2FSineRegressionOnTIs.pdf&ei=6vnSUtD7MabQsATKzIGADA&usg=AFQjCNF-sJS1KCsu2KCzIYldLGyPhBob\\_g&sig2=Vv9gKOJQoVF0ix1eNCE8NA&bvm=bv.59026428,d.cWc\)](http://www.google.com/url?sa=t&rct=j&q=&esrc=s&source=web&cd=2&ved=0CDkQFjAB&url=http%3A%2F%2Fwww.austincc.edu%2Fjthom%2FSineRegressionOnTIs.pdf&ei=6vnSUtD7MabQsATKzIGADA&usg=AFQjCNF-sJS1KCsu2KCzIYldLGyPhBob_g&sig2=Vv9gKOJQoVF0ix1eNCE8NA&bvm=bv.59026428,d.cWc)

**Example:** Use your calculator to model the circadian rhythm in the last example.

**Example:** Here is a cosine to model. These are average temperatures for Montgomery County, Maryland. Use Month = 0 for January for easier modeling. \* *Precalculus: Mathematics for Calculus*, Stewart, 5<sup>th</sup> ed., p. 463

Month	Ave Temp °F	Month	Ave Temp °F
January	40.0	July	85.8
February	43.1	August	83.9
March	54.6	September	76.9
April	64.2	October	66.8
May	73.8	November	55.5
June	81.8	December	44.5