§8.4 Plane Curves & Parametric Equations (Supplemental by Stewart) Further Supplement from Handouts §6.4 & 10.7 by Rockswold, Lial & Hornsby

Plane curves show a point moving in a plane. In other words a plane curve has ordered pairs as well as a direction of movement as t, time, increases.

Sketching a Plane Curve

1) Find x & y coordinates for parametric equations over an interval containing both positive & negative values of t.
2) Plot (x, y)'s in the rectangular coordinate system
3) Draw arrows to represent the direction of movement as t increased.

Example: Sketch $x = 2t + 1$ $y = (t + {}^{1}/_{2})^2$
#4a p. 807 from Ed. 5 of Stewart's Precalculus

Note: *I used* $-{}^{7}/_{2} \leq t \leq {}^{5}/_{2}$ *by increments of* ${}^{1}/_{2}$*'s.*

Next, we want to take an equation from its parametric form to its equivalent rectangular form by **eliminating a parameter.**

Steps for Writing a Parametric Equation as a Linear Equation

1) Solve the simpler equation for the parameter t
2) Substitute into the other equation
3) Simplify

Example: Give the linear form for $x = 2t + 1$ & $y = (t + {}^{1}/_{2})^2$

Note: *The equation we obtain is a parabola and notice how the points that we obtained before conform to the equation.*

What if the equations are trigonometric in type? Then we will follow the following procedures

Steps for Parametric Equations as Trig Functions

1) Find an "identity" that relates the functions
2) Substitute the rectangular coordinate values in for the trig function in step 1
3) Simplify
4) Make sure the range of the trig function(s) are notes (a graph of the parametric can help here)

Example: Give the linear form of $x = \cot t$ & $y = \csc t$ on $0 < t < \pi$

*#18 p. 807 Stewart's Precalculus Ed. 5

Creating the Parametric Equation for a Line

1) Know or find the slope of the line
2) x = x-coordinate + t (use denominator of slope x t)
3) y = y-coordinate + mt (Use slope • t; slope written as unit rate)

Example: Find the parametric equation for a line thru (12, 7) & the origin

*#26 p. 807 Stewart's Precalculus Ed. 5

There are two types of curves described by parametric equations that are worth note:

Cycloid The position of a point P on a circle of radius, r, as it is rolled along the x-axis.

$$x = a(\theta - \sin\theta) \quad \& \quad y = a(1 - \cos\theta)$$

Where a = radius & θ = angle of the arc subtended in the roll

Lissajous Figure These are "figure 8" type figures described by

$$x = A \sin \omega_1 t \qquad y = B \cos \omega_2 t$$

A & B can be used to find the width & height of the figure ($-A \le x \le A$ & $-B \le y \le B$) and ω_1 tells us how many loops vertically & ω_2 tells us how many horizontal loops.

Let's try graphing a Lissajous Figure with our calculator

Example: On your TI graph x = sin 4t & y = cos t

*#40 p. 807 Stewart's Precalculus Ed. 5

Step 1: MODE & set to radians & PAR

Step 2: Y = X_{1T} = sin (4t t using xTθn key
Y_{1T} = cos (t

Step 3: WINDOW Leave t min, tmax & tstep alone
Set x min = -1, x max = 1, x scl = 0.1
y min = -1, y max = 1, x scl = 0.1

Step 4: GRAPH

Now alternate, make y = cos 3t while x = sin t and then try to do both together with ω's.

Our last task is to express a Polar Equation in Parametric Form

> **Parametric Form of a Polar Eq.**
>
> If $r = f(\theta)$ is in polar form, then
>
> $x = f(t)\cos t$ & $y = f(t)\sin t$
>
> Is the parametric form of the polar equation

How To Proceed:

1) Solve $r = f(\theta)$ for θ by finding $f^{-1}(r) = \theta$
2) Let $t = \theta$ in original r getting $r = f(t)$
3) Use $x = r\cos\theta$ & $y = r\sin\theta$
   Substitute r's value with f(t) [see 2) above]
4) Now
   Substitute θ with $f^{-1}(r)$ [see 1) above]
5) Then
   Substitute $r = f(t)$ [see 2) above]

Let me show a quick easy example that we should recognize as a spiral from our polar graph study.

Example: $r = 3\theta$

1) Solve $r = 3\theta$ gives us $\theta = {}^{r}/_{3}$
2) Replace θ with t in original polar eq. ➔ $r = 3t$
3) Replacing r with 3t
   $x = r\cos\theta$ ➔ $x = 3t\cos(\theta)$
   $y = r\sin\theta$ ➔ $y = 3t\sin(\theta)$
4) Replacing θ with ${}^{r}/_{3}$
   $x = 3t\cos(\theta)$ ➔ $x = 3t\cos({}^{r}/_{3})$
   $y = 3t\sin(\theta)$ ➔ $y = 3t\sin({}^{r}/_{3})$
5) Replacing r with 3t
   $x = 3t\cos({}^{r}/_{3})$ ➔ $x = 3t\cos({}^{3t}/_{3})$ ➔ $x = 3t\cos t$
   $y = 3t\sin({}^{r}/_{3})$ ➔ $y = 3t\sin({}^{3t}/_{3})$ ➔ $y = 3t\sin t$

Example: $r = 5 \sin \theta$

1) Solve $r = 5 \sin \theta$ for θ

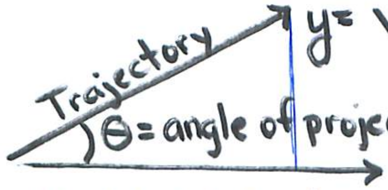
2) Replace θ with t in $r = 5 \sin \theta$

3) Replace r w/ 2)'s value in $x = r \cos \theta$ & $y = r \sin \theta$

4) Replace θ w/ 1)'s value in 3)'s

5) Replace r in 4) with 1)'s value & simplify to finish

Modeling Projectile Motion w/ Parametric Equations



$y = v_0 \sin\theta \cdot t - 16t^2$

The vertical component adjusted for gravity. Even in the rectangular equation this y value will represent the height from the ground.

$x = v_0 \cos\theta \cdot t$

The horizontal component. The distance traveled horizontally, even in the rectangular eq.

Solving: Make an equation for rectangular coordinate system

1) Solve x for t
2) Substitute t value into y & simplify
   The result will be a parabola's equation, as we would expect.

Finding Maximum Height

1) Find vertex of rectangular equation
   $x = -b/2a \qquad y = f(-b/2a)$ or $-\dfrac{b^2 - 4ac}{4a} = \dfrac{4ac - b^2}{4a}$
2) The y-value is the height
   *Note: The x-value is the horizontal distance that the projectile has traveled when reaching its maximum height.
3) If time is needed to reach that height calculate x & substitute it into $t = \dfrac{x}{v_0 \cos\theta}$

Finding Distance & Time 'til hits Ground

1) Find the x-intercept of the rectangular equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This is the horizontal distance when the projectile hits the ground. The vertical is zero at that point!

2) If you need time, again take value of x & plug back into

$$t = \frac{x}{v_0 \cos \theta}$$

Finding Height or Distance @ Given Vertical

1) Plug in vertical for y in rectangular equation

2) Set equal to zero (move y-value to other side.

3) Repeat quadratic formula as above. This time you may get a neg. value that is extraneous.

Finding Height or Horizontal @ Give t

1) Find x by plugging into

$$x = v_0 \cos\theta \cdot t$$

This is horizontal distance

2) Find y by plugging into

$$y = v_0 \sin\theta \cdot t - 16t^2$$

This is vertical height