Practice Final

(17) In the following list of function, all have a local min at $x=0$. The 2^{nd} Derivative Test can show this for some but not others. List those that can.

$y=\sqrt{x}$ $\quad y'=\frac{1}{2\sqrt{x}}$ $\quad y''=-\frac{1}{4x^{3/2}}$ No $\quad y''$ is undefined at $x=0$.

$y=\sqrt[3]{x^2}$ $\quad y'=\frac{2}{3}x^{-1/3}$ $\quad y''=-\frac{2}{9}x^{-4/3}$ No $\quad y''$ is undefined at $x=0$

$y=|x|$ $\quad$ derivative dne at zero there's a corner

$\boxed{y=-e^{-x^2}}$ $\quad y'=2xe^{-x^2}$ $\quad y''=2e^{-x^2}+-4x^2e^{-x^2}$ Yes

$\boxed{y=-\cos x}$ $\quad y'=\sin x$ $\quad y''=\cos x$ $\quad$ Yes

(18) If the second derivative of a f(n) exists at $x=a$, then f(n) is continuous at "a"

$\boxed{\text{This is a true statement}}$ $\quad$ Review definition of continuous f(n)

(19) Assess the truth of each statement

ⓐ There are some functions with no critical pts. $\quad$ ex. $\quad$ $\boxed{\text{true}}$

ⓑ There are some f(n) w/ an infinite # of critical points $\quad$ $\boxed{\text{true}}$

ⓒ Every cubic polynomial has ≥ 1 critical pt. $\quad$ $\boxed{\text{False}}$

ⓓ Every function defined on a closed interval $[a,b]$ has at least 2 critical pts. $\quad$ $\boxed{\text{True}}$

ⓔ All local extrema are critical pts of a f(n) $\quad$ $\boxed{\text{True}}$

(20) For the f(n) $y=-x^2+bx+3$, which statement best describes b's role in the function.

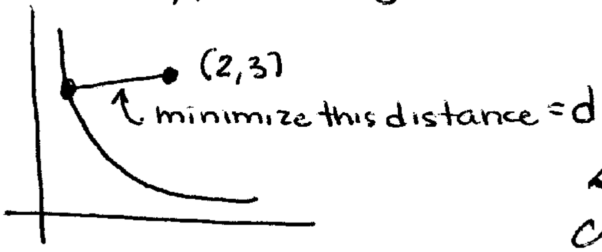
$f'(x)=-2x+b$ is slope so $b=$ slope at $x=0$

ⓐ b determines $f(0)$ | ⓑ $\boxed{\text{b determines m @ } x=0}$ | ⓒ b determines concavity @ $x=0$

ⓓ b determines concavity as $x\to\pm\infty$ $\quad$ ⓔ b determines sign of f $x\to\pm\infty$

Practice Final

(21) Use your calculator to find the coordinates of the point on the hyperbola $f(x) = 1/x$ that is closest to (2,3)



$d = \sqrt{(3 - 1/x)^2 + (2-x)^2}$

so the minimum is a critical point ∴ find $\frac{dd}{dx}$ or d'

$$d' = \frac{2x^4 - 4x^3 + 6x - 2}{2x^3\sqrt{(3-1/x)^2 + (2-x)^2}}$$

after a very messy differentiation & with simplification

I recommend plugging this into your calculator y=? and the use 2nd [TRACE] to find the intersection w/ the x-axis ([zero]). If you've zoomed your calculator into the right side of the graph [xmin=0, xmax=5, ymin=-10 y-max=10] you will more easily locate a left bound & right bound [point to left of zero & to right of zero] in which to approx. (guess) & then calculate the x-intercept $x = 0.358\,5549$ & $y = 0$. If you plug $x \approx 0.359$ into $f(x) = 1/0.359 \approx 2.79$

Thus the point closest to (2,3) on the curve is $\boxed{(0.359,\ 2.79)}$

(22) The Mean Value Theorem states that if $f(x) = 1/x$ is continuous on [1,5] and differentiable on (1,5) then there is a c with $1 < c < 5$ so that $f'(c) = f(5) - f(1) / 5 - 1$. Find the value of c for this situation.

$f'(c) = \frac{1/5 - 1}{4} = -\frac{1}{5}$ or -0.2 ∴ $m = -0.2$ & $f'(x) = \frac{-1}{x^2}$

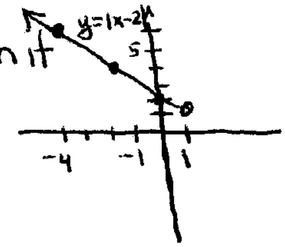
so $\frac{-1}{x^2} = \frac{-1}{5}$ and thus $x^2 = 5 \Rightarrow x = \sqrt{5} \approx \boxed{2.24}$

Practice Final

(23) $f(x) = \begin{cases} |x-2| & x<1 \\ x^2+C & x \geq 1 \end{cases}$

What value makes the function continuous at C?

Graph it



$\boxed{C=0}$ will make the function continuous

Therefore x^2+C must be 1 at x=1 to be continuous

so solve $x^2+C=1$

so $C=1-x^2$

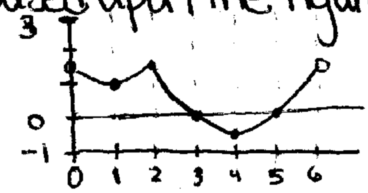
& x=1 so $C=1-1=0$

(24) What value of C in #23 will make the f(n) differentiable at x=1?

The function will never be differentiable at 1, b/c the $\lim\limits_{x\to 1^-} \neq \lim\limits_{x\to 1^+}$

(25) Based upon the figure A

~30



(25) Figure A represents f'. How many local mins are on $0<x<6$?

$\boxed{\text{one; } x=5}$ $f'(x)<0$ to it's left & $f'(x)>0$ to right

(26) Figure A represents f'. How many inflection points are in f(x) on $0<x<6$?

$\boxed{\text{three; } x=1,2,4}$ Every time the slope changes from decreasing to increasing or vice versa is an inflection point (the critical pts of f')

(27) Fig. A shows a function, g now. What is the average rate of change over [1,4]

ave rate of change $= \frac{g(4)-g(1)}{4-1} = \frac{-0.5-1}{3} = \frac{-1.5}{3}$

$= \boxed{-0.5}$ it's actually -0.53

(28) Fig A shows a function, h. Find F'(1)

for $F(x) = x^2 \cdot h(x) \Rightarrow F'(x) = 2x\,h(x) + x^2 h'(x)$

when $x=1 \Rightarrow h'(1)=0,\ h(1)=1$ so

$F'(x) = 2(1)(1) + (1)^2 \cdot 0 = \boxed{2}$

Practice Final

(29) Fig. A now shows j. Can the 2^{nd} Derivative test be used to show that $j(x)$ has a local max at $x=2$?

$\boxed{\text{No, there is no derivative a } x=2\text{, it's a corner}}$

(30) Fig. A is k. Find the best estimate of $k'(3)$.

The value of $k'(3)$ is the slope of the tangent line at $x=3$. Inspection shows such a line to go through $(0,3)$ & $(-1,4)$ & thus the resultant slope is $\boxed{m=-1}$

(31) The fcn $f(x)=x^x$ where $x>0$ is neither exponential nor a power function, but by using ln and implicit differentiation it can be differentiated. ~~Belo~~ Find an estimate correct to one decimal for the value of $f''(2)$.

$$\ln y = x \ln x \quad \therefore \quad \frac{1}{y} y' = \ln x + x \cdot \frac{1}{x}$$

$$\Rightarrow y' = y(\ln x + 1) = y\ln x + y$$

$$\therefore y'' = y \cdot \frac{1}{x} + y'\ln x + y'$$

& since $y'\big|_{x=2} = (\ln 2 + 1)[2^2]$ This is $y=x^x$

$$y''\big|_{x=2} = 4 \cdot \tfrac{1}{2} + 4(\ln 2+1) + 4(\ln 2+1)(\ln 2)$$

or

$$y''\big|_{x=2} = 4(\ln 2+1)[\ln 2+1] + 4 \cdot \tfrac{1}{2}$$

$$13.4669895 \approx \boxed{13.5}$$

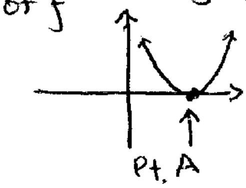
(32) $f(x) = \ln x$ & $g(x) = x-1$ are continuous on $[1,5]$ & differentiable on $(1,5)$

$f'(x) = \frac{1}{x}$ & $g'(x) = 1$ $\therefore$ on $[1,5]$ $f'(x) \le g'(x)$ and the Racetrack Principle can $\therefore$ be used to show that $\boxed{f(x) \le g(x) \text{ on } [1,5]}$

Practice Final Key Con'd

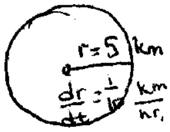
(33) An inflection pt. of f can't also be a c.p. of f. $\boxed{\text{False}}$

Counterexample: Think of the graph of f'



Pt A on f' is both a cp. since $f'(x)=0$ and an inflection point since f' is at a minimum there.

(36) Suppose that an oil spill spreads out in the shape of a circle on the sea's surface. If the radius of the oil slick is increasing at a rate of 1/10 km per hour when the radius is 5km, at what rate, in km²/hr, is the area increasing at this time?



$A=\pi r^2 \Rightarrow A' = 2\pi r \frac{dr}{dt} \Rightarrow A' = 2\pi(5)(\frac{1}{10}) = \pi \frac{km^2}{hr}$

so the area is increasing at a rate of $\boxed{\pi \frac{km^2}{hr}}$

(37) Consider line L_1, tangent to $f(x)=x^2$ at $x=2$ and line L_2, tangent to $f(x)=x^2$ at $x=-3$. Where in the xy-plane do L_1 & L_2 intersect?

$f(x)=x^2$ so $f'(x)=2x$ & $\therefore$ m @ $x=2$ is $m_1 = 2(2)=4$

& eq. of line tangent to curve is

$y-(2)^2 = 4(x-2) \Rightarrow y=4x-8+4$ & $y=4x-4$

$y=f(x)=(x)^2$

Now the point the tangent lines meet is their point of intersection (sol. to system)

$\therefore$ m @ $x=-3$ is $m_2 = 2(-3)=-6$

$y-(-3)^2 = -6(x-(-3)) \Rightarrow y=-6x-18+9 \Rightarrow y=-6x+-9$

$4x-4=-6x-9 \Rightarrow 10x=-5 \Rightarrow x=\frac{-1}{2}$

& $\therefore$ $y=4(\frac{-1}{2})-4=-2+-4=-6$

The point the lines meet is $\boxed{(-0.5,-6)}$

(38) If $h(x)=\frac{f(x)}{g(2x)}$, find $h'(1)$

$h'(x)=\frac{f'(x)g(2x)-f(x)g'(2x)\cdot 2}{[g(2x)]^2}$

$f'(1)=-4$; $g(2\cdot1)=g(2)=-2$; $f(1)=3$

$g'(2\cdot1)=g'(2)=8$;

$\therefore$ $h'(1)=\frac{-4(-2)-3(8)(2)}{(-2)^2}=\frac{8-24\cdot2}{4}=\frac{-40}{4}=\boxed{-10}$

(39) If $j(x)=f(g(x))$, find $j'(3)$

$j'(x)=f'(g(x))\cdot g'(x)$

$g(3)=1$; $f'(1)=-4$; $g'(3)=-2$ $\therefore$ $j'(3)=(-4)(-2)=\boxed{8}$

Practice Final Key con'd

(40) If $k(x) = [f(x)]^3$, find the value of $k'(3)$ $\quad k'(x) = 3[f(x)]^2 \cdot f'(x)$

$f(3) = -1$; $f'(3) = 2$ $\quad \therefore\ k'(3) = 3[-1]^2 \cdot (2) = 3 \cdot 1 \cdot 2 = \boxed{6}$

(41) " Given the fcn) the one with the greatest derivative at $x=0$ is...

$f(x) = 2^x \quad f'(x) = 2^x \ln 2 \quad f'(0) \doteq 0.693$

$f(x) = \tan x \quad f'(x) = \sec^2 x \quad f'(0) = 1$ $\quad$ *Remember* $\sec = \frac{1}{\cos}$, so $\left(\frac{1}{1}\right)^2$

$f(x) = x^2 \quad f'(x) = 2x \quad f'(0) = 0$

$f(x) = \sin \frac{x}{2} \quad f'(x) = \frac{1}{2}\cos\frac{x}{2} \quad f'(0) = 0.5$

$f(x) = \frac{1}{x+2} \quad f'(x) = \frac{-1}{(x+2)^2} \quad f'(0) = \frac{-1}{2^2} = \frac{-1}{4} = -0.25$

Greatest is $\boxed{f(x) = \tan x}$

(42) The first derivative of $f(x) = a^{x+b}$, $a > 0$ is...

$f'(x) = a^{x+b} \ln a$ $\quad \boxed{\text{can be either } +/-/0}$ since when $a < 1$ it is neg

if $a = 1$ it would be zero

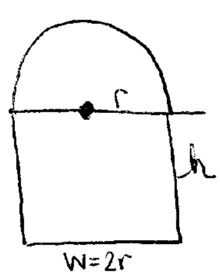
if $a > 0$ it is positive

(43) Consider $f(x) = x^3 - x$ Over which interval is $f''(x) < 0$?

$f'(x) = 3x^2 - 1$ and $f''(x) = 6x$ $\quad \therefore \quad 6x < 0$ when $x < 0$

$\therefore\ \boxed{(-\infty, 0)}$ is the interval on which $f''(x) < 0$

(44)



$$P = 1 = \underbrace{2r + 2h}_{\text{rectangle}} + \underbrace{\tfrac{1}{2}(2\pi r)}_{\text{semicircle}} \quad \& \quad A = \overbrace{h \cdot w}^{\text{rectangle}} + \overbrace{\tfrac{1}{2}\pi r^2}^{\text{semicircle}}$$

$$= 2rh + \tfrac{1}{2}\pi r^2$$

Solving P for h we find $h = \frac{1 - 2r - \pi r}{2}$

Now we have $A = \left(\frac{1 - 2r - \pi r}{2}\right) \cdot 2r + \frac{1}{2}\pi r^2$

$$= r - 2r^2 - \pi r^2 + \tfrac{1}{2}\pi r^2 = r - \left(\tfrac{4+\pi}{2}\right) r^2$$

$-\frac{4r^2}{2} - \frac{2\pi r^2}{2} + \frac{1\pi r^2}{2} = \frac{-4r^2 - \pi r^2}{2} = -r^2\left(\frac{4+\pi}{2}\right)$

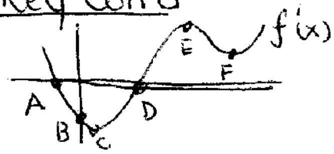
$A' = 1 - (4+\pi)r$ so when $A' = 0 \Rightarrow (4+\pi)r = 1 \Rightarrow r = \frac{1}{4+\pi}$

$A = h \cdot W + \frac{1}{2}\pi r^2 \Rightarrow A = \frac{1 - 2\left(\frac{1}{4+\pi}\right) - \pi\left(\frac{1}{4+\pi}\right)}{\cancel{2}} \cdot \cancel{2}\left(\frac{1}{4+\pi}\right) + \frac{1}{2} \cdot \pi \cdot \left(\frac{1}{\pi+4}\right)^2 = \frac{4+\pi-2-\pi}{(4+\pi)^2} + \frac{\pi}{2(\pi+4)^2}$

$\frac{4+\pi}{4+\pi}$

$\Rightarrow \quad A = \frac{2}{(\pi+4)^2} + \frac{\pi}{2(\pi+4)^2} = \frac{4+\pi}{2(\pi+4)^2} = \frac{1}{2(\pi+4)} \approx \boxed{0.07}$

Practice Final Key Con'd

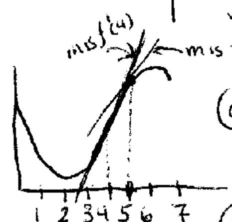
(45) On the graph 

ⓐ Inflection points are max/min of f' so $\boxed{C, E \,\&\, F}$ are T.P.

ⓑ Critical points are where f(x) crossed x-axis so $\boxed{A \,\&\, D}$ are C.P.

ⓒ A minimum is when slopes go from neg to positive so $\boxed{D}$ is a min

ⓓ A maximusm is when slopes go from pos to neg so $\boxed{A}$ is a max

(46) For the diagram 

ⓐ The slope of tangent lines is derivative at pt $\boxed{f'(4) > f'(5)}$ ∴

ⓑ Second derivative indicates concavity
at $x=1$ the curve is concave up so $f''(1)>0$
at $x=4$ the curve is concave down or it is an inflection point so $f''(4) \leq 0$

∴ $\boxed{f''(1) > f''(4)}$

(47) Find the derivatives ⓐ $y=\sqrt{x^2+3} = (x^2+3)^{1/2} \Rightarrow \boxed{\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2+3}} = \frac{x}{\sqrt{x^2+3}}}$

ⓑ $y = x(x+a)^7 \Rightarrow y' = (x+a)^7 + 7x(x+a)^6 = (x+a)^6[x+a+7x] = \boxed{(x+a)^6(8x+a)}$

ⓒ $t = x\ln x \Rightarrow \frac{dt}{dx} = \ln x + \frac{x}{x} = \boxed{\ln x + 1}$

ⓓ $w = \frac{x-2}{x^2+8} \Rightarrow w' = \frac{1(x^2+8) - \overset{-2x^2+4x}{2x(x-2)}}{(x^2+8)^2} = \frac{-2x^2+x^2+4x+8}{(x^2+8)^2} = \frac{-x^2+4x+8}{(x^2+8)^2}$

ⓔ $y = e^{(e^x+4)} \Rightarrow y' = e^x e^{e^x+4} = \boxed{e^{e^x+x+4}}$

ⓕ $f(x) = 4x^3-5x^2+4x+10 \Rightarrow f'(x) = 12x^2-10x+4 = \boxed{2(6x^2-5x+2)}$

ⓖ $f(x) = \frac{3}{x} + \sqrt{9x+1} \Rightarrow \boxed{f'(x) = \frac{-3}{x^2} + \frac{9}{2\sqrt{9x+1}}}$

Practice Final Key Con'd

(48) A water park finds at an admission price of \$17, attendance is 450 per day. For every \$1 decrease in price, 30 more people visit the park/day. What is the park attendance when admission prices are set to maximize revenue?

$R = p \cdot q \Rightarrow p = 17 - \frac{1}{30}(q - 450)$ where p = price & q = # people

$\therefore p = 17 - \frac{1}{30}q + \cancel{\$}15 = -\frac{1}{30}q + 32$ so $R(q) = q\left(-\frac{1}{30}q + 32\right) = -\frac{1}{30}q^2 + 32q$

To maximize revenue $R'(q) = 0 \Rightarrow -\frac{2}{30}q + 32 = 0 \Rightarrow \frac{1}{15}q = 32 \Rightarrow q = 480$

Park attendance is $\boxed{480 \text{ people}}$ when price is set to max. attend.

(49) Find the following & graph $f(x) = -2x^3 + 3x^2 + 12x$

(a) C.P. w/ Calculus & give o.p.'s $f'(x) = -6x^2 + 6x + 12 = 0 \Rightarrow -6(x^2 - x - 2) = 0$

$\Rightarrow -6(x-2)(x+1) = 0 \Rightarrow$ $x - 2 = 0$, $x = 2$; $x + 1 = 0$, $x = -1$

$f(2) = -2(2)^3 + 3(2)^2 + 12(2) = 20$ & $f(-1) = -2(-1)^3 + 3(-1)^2 + 12(-1) = -7$

$\therefore \boxed{(2, 20)}$ $\boxed{(-1, -7)}$

(b) Max/Min based on 2nd Derivative Test

$f''(x) = -12x + 6 \Rightarrow f''(2) = -12(2) + 6 < 0 \therefore$ concave down & a $\boxed{\text{max @ } x = 2}$

$f''(-1) = -12(-1) + 6 > 0 \therefore$ concave up & a $\boxed{\text{min @ } x = -1}$

Note: $f'(1) = -6(1)^2 + 6(1) + 12 > 0$ & $f'(3) = -6(3)^2 + 6(3) + 12 < 0$ $\nearrow \searrow$ 1st derivative test also shows max at $x = 2$

$f'(-2) = \overset{-24}{-6(-2)^2} \overset{+ -12}{+ 6(-2)} \overset{+12}{+ 12} < 0$ & $f'(0) = -6(0)^2 + 6(0) + 12 > 0$ $\searrow \nearrow$ 1st derivative test also shows min at $x = -1$

(c) I.P. & give as o.p.'s $f''(x) = -12x + 6 = 0 \Rightarrow 12x = 6 \Rightarrow \boxed{x = \frac{1}{2}}$

$f\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) = 6.5 \therefore \boxed{\left(\frac{1}{2}, 6.5\right)}$

(d) Show $x = \frac{1}{2}$ is a I.P. w/ 2nd Derivative $f''(0) = -12(0) + 6 > 0$ concave up to left of $\frac{1}{2}$

up down $x = \frac{1}{2}$

$f''(1) = -12(1) + 6 < 0$ concave down to right of $\frac{1}{2}$

(e) Y-int as op. $f(0) = -2(0)^3 + 3(0)^2 + 12(0) = 0$

$\boxed{(0, 0)}$

Practice Final Key Con'd

(49) (f) X-int as $\approx$ op. $f(x)=0$ is x-int so $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ will give intercept(s)

$$x = \frac{-(3) \pm \sqrt{3^2-4(-2)(12)}}{2(-2)} = \frac{-3 \pm \sqrt{105}}{-4} \doteq \frac{-3 \pm 10.2}{-4} \text{ so } x \doteq \frac{-3+10.2}{-4} \approx -1.8$$

$$\text{or } x \doteq \frac{-3-10.2}{-4} \approx 3.3$$

$\boxed{(-1.8, 0) \text{ \& } (3.3, 0)}$ are approx. x-int.

(g) See the next page for the graph.

(50) Find the antiderivative of each

(a) $f(x) = x^2 + 5 \Rightarrow \boxed{F(x) = \frac{x^3}{3} + 5x + C}$

* (b) $f(x) = \frac{5e^x}{5+e^x}$ * Oops sorry too difficult for us!

Let $u = 5+e^x$ $du = e^x dx$

so $\int f(x) = 5 \cdot \frac{1}{u} \cdot \frac{du}{dx}$ $\therefore$ $F(x) = 5 \ln u + C$

& thus $\boxed{F(x) = 5 \ln(5+e^x) + C}$

(c) $f(x) = x^2(2x-1) = 2x^3 - x^2$

$\boxed{F(x) = \frac{x^4}{2} - \frac{x^3}{3} + C}$

(51) Find F(x) for $f(x) = x^2 - \frac{4}{x} + \frac{8}{x^3}$

$\Rightarrow \boxed{F(x) = \frac{x^3}{3} - 4\ln|x| - \frac{4}{x^2} + C}$

(52) Find G(z) with $G'(z) = g(z)$ and $G(0) = 4$ given that $g(z) = z - \sqrt{z}$

$G(z) = \frac{z^2}{2} - \frac{2z^{3/2}}{3}$ $G(0) = \frac{0^2}{2} - \frac{2(0)^{3/2}}{3} + C = 4 \Rightarrow C = 4$

$\therefore \boxed{G(z) = \frac{z^2}{2} - \frac{2z^{3/2}}{3} + 4}$

(53) Find the horizontal asymptote(s) of $f(x) = \frac{x^3+3x}{1-x^4}$

$$\lim_{x\to\pm\infty} \frac{x^3+3x}{1-x^4} = \lim_{x\to\infty} \frac{x^3/x^4 + 3x/x^4}{1/x^4 - 1} = \lim_{x\to\infty} \frac{\cancel{1/x}^{\,0} - \cancel{3/x^3}^{\,0}}{\cancel{1/x^4}^{\,0} - 1} = \frac{0}{-1} = 0$$

$\therefore$ $y = 0$ is a horizontal asymptote as $x \to \infty$ or $x \to -\infty$

(54) Find the vertical asymptote(s) of $f(x) = \frac{x^2-4}{x^2-3x-10}$

$f(x) = \frac{(x-2)\cancel{(x+2)}}{(x-5)\cancel{(x+2)}}$ so $\lim_{x\to5^+} \frac{x^2-4}{x^2-3x-10}$ so $x = 5.01$ $\frac{21.1001}{0.0701} = 301$ So $x = 5.0001$ $\frac{21.00100001}{0.00070001} = 30{,}001$

$x = 5.001$ $\frac{21.010001}{0.007001} = 3001$ $\to \infty$

$\boxed{\begin{array}{l} \text{As } x \to 5^+ \quad f(x) \to \infty \\ \quad x \to 5^- \quad f(x) \to -\infty \end{array}}$

$\lim_{x\to5^-} \frac{x^2-4}{x^2-3x-10}$ so $x = 4.99$ $\frac{20.9001}{-0.0699} = -299$

$x = 4.999$ $\frac{20.990001}{-0.006999} = -2999$ $\to -\infty$