Key for Midterm #3a & #3b

#1a

a) $f(x) = \ln(\cos^{-1} x) \Rightarrow f'(x) = \frac{1}{\cos^{-1} x} \cdot \frac{-1}{\sqrt{1-x^2}} = \boxed{\frac{-1}{(\cos^{-1} x)\sqrt{1-x^2}}}$

b) $f(x) = (\sin x)^{-1}$; $f'(x) = -1(\cos x)(\sin x)^{-2} = \frac{-\cos x}{\sin x} \cdot \frac{1}{\sin} = \boxed{-\cot x \csc x}$

or b) $y' = \frac{d}{dx}\cot x \sec x + \cot x \frac{d}{dx} \sec x = -\csc^2 x \cdot \sec x + \cot x \cdot \sec x \cdot \tan x$ Note: $\cot x \cdot \tan x = \frac{1}{\tan x} \cdot \tan x = 1$

$= -\csc^2 x \sec x + \sec x = \sec x(1 - \csc^2 x) = -\cot^2 x \sec x = \frac{-\cos x}{\sin x} \cdot \frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cancel{\cos x}}$

$= \boxed{-\cot x \csc x}$

c) $\boxed{g'(z) = \frac{5^x \ln 5}{\sqrt{1-5^{2x}}}}$

d) $F(x) = \frac{3 \log_2 x}{4} = \frac{3}{4} \cdot \log_2 x$; $F'(x) = \frac{3}{4} \cdot \frac{1}{x \ln 2} = \boxed{\frac{3}{4x \ln 2}}$ or $F'(x) = \frac{1}{4x^3 \ln 2} \cdot 3x^2 = \boxed{\frac{3}{4x \ln 2}}$

#1b

a) $g'(x) = \frac{1}{\sin^{-1}(x)} \cdot \frac{1}{\sqrt{1-x^2}} = \boxed{\frac{1}{\sin^{-1} x \sqrt{1-x^2}}}$

b) $f(x) = (\cos x)^{-1}$; $f'(x) = -1(\cos x)^{-2} \cdot -\sin x = \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \boxed{\tan x \sec x}$

$\cot x \cdot \tan x = 1$

or $y' = \sec^2 x \csc x + -\cot x \csc x \tan x = \sec^2 x \csc x - \csc x = \csc x(\sec^2 x - 1)$

$= \csc x (\tan^2 x) = \frac{1}{\sin x} \cdot \frac{\sin x}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \tan x}$

e) $h'(t) = \boxed{\frac{-3^x \ln 3}{\sqrt{1-3^{2x}}}}$

d) $F'(x) = \frac{4}{5} \cdot \frac{1}{x \ln 3} = \boxed{\frac{4}{5x \ln 3}}$ or $F'(x) = \frac{1}{5} \cdot \frac{1}{x^4 \ln 3} \cdot 4x^3 = \boxed{\frac{4}{5x \ln 3}}$

#2a $\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)^3 = 3\left(\frac{\sin x}{\cos x}\right)^2 \cdot \frac{\overbrace{\cos x \cos x + \sin x \sin x}^{\cos^2 x + \sin^2 x = 1}}{\cos^2 x} = \boxed{3 \tan^2 x \sec^2 x}$

#2b $\frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)^3 = 3\left(\frac{\cos x}{\sin x}\right)^2 \cdot \frac{-\sin x \sin x - \cos x \cos x}{\sin^2 x} = \boxed{-3 \cot^2 x \csc^2 x}$

$-(\sin^2 x + \cos^2 x) = -1$

#3a

$y = (\sin^{-1} x)^2 \Rightarrow \sqrt{y} = \sin^{-1} x \Rightarrow \sin\sqrt{y} = \sin(\sin^{-1} x) \Rightarrow \sin\sqrt{y} = x$

by implicit differentiation of $\sin\sqrt{y} = x \Rightarrow \frac{\cos\sqrt{y}}{2\sqrt{y}} \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{2\sqrt{y}}{\cos\sqrt{y}}$ ← substitute $\sqrt{y} = \sin^{-1} x$

& since $\cos^2\sqrt{y} + \sin^2\sqrt{y} = 1 \Rightarrow \therefore \cos\sqrt{y} = \sqrt{1-\sin^2\sqrt{y}}$

$\frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-\sin^2\sqrt{y}}}$ ← $x = \sin\sqrt{y}$ ∴ $x^2 = \sin^2\sqrt{y}$ $\Rightarrow \boxed{\frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}}$

#3b

$y = (\cos^{-1} x)^2 \Rightarrow \sqrt{y} = \cos^{-1} x \Rightarrow \cos\sqrt{y} = \cos(\cos^{-1} x) \Rightarrow \cos\sqrt{y} = x$

by implicit differentiation of $\cos\sqrt{y} = x \Rightarrow \frac{-\sin\sqrt{y}}{2\sqrt{y}} \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{y}}{\sin\sqrt{y}}$ ← substitute $\sqrt{y} = \cos^{-1} x$

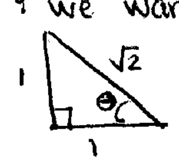
& $\sin^2\sqrt{y} + \cos^2\sqrt{y} = 1$ ∴ $\sin\sqrt{y} = \sqrt{1-\cos^2\sqrt{y}}$ ← $x = \cos\sqrt{y}$ so $x^2 = \cos^2\sqrt{y}$

$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-2\cos^{-1} x}{\sqrt{1-x^2}}}$

#4a&b $\tan^{-1}(\cos \pi)$

$\cos \pi = -1$ from the unit circle

& we want the angle that yields a -1 for $\tan\theta$ on $-\pi/2 < \theta < \frac{\pi}{2}$



where $\theta = \pi/4$ or $45°$ $\tan\theta = 1$ & $\tan\theta = -1$

either in QII at 135° or QIV at $-45°$

$\therefore \tan^{-1}(\cos \pi) = -\pi/4$

On test 3a) ii) $-\pi/4$

on test 3b vii) $-\pi/4$

Key for Midterm #3a & 3b Cond

#5a & 5b 1st derivative $3x^2 + 3y^2\frac{dy}{dx} = 0$ Solve $\Rightarrow \frac{dy}{dx} = \frac{-3x^2}{3y^2} = \frac{-x^2}{y^2}$

2nd derivative $\frac{d^2y}{dx^2} = \frac{-2xy^2 + 2x^2\frac{dy}{dx}}{y^4} = \frac{-2xy^2 + 2x^2y\left(\frac{-x^2}{y^2}\right)}{y^4}$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{-2xy^3}{y} + \frac{2x^4}{y}}{y^4} = \frac{-2x(y^3+x^3)}{y^5} = \frac{-2x}{y^5}$ → orig. eq. = 1

on test a) i) $\frac{-2x}{y^5}$
on test b) iii) $\frac{-2x}{y^5}$

#6a & 6b 1st derivative $2x + 3\cdot 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-2x}{6y} = \frac{-x}{3y}$

on test a) m @ $(\frac{1}{2},\frac{1}{2}) \Rightarrow \frac{dy}{dx} = \frac{-2(\frac{1}{2})}{6(\frac{1}{2})} = \frac{-1}{3}$

$\therefore y - \frac{1}{2} = -\frac{1}{3}(x - \frac{1}{2}) \Rightarrow y - \frac{1}{2} = -\frac{1}{3}x + \frac{1}{6} \Rightarrow y = -\frac{1}{3}x + \frac{2}{3}$ ($+\frac{3}{6}$, $+\frac{3}{6} = \frac{4}{6} = \frac{2}{3}$)

on test a) & $\therefore$ y-int. $\boxed{(0, \frac{2}{3})}$

on test b) m @ $(\frac{1}{3}, 4)$ is $\frac{-\frac{1}{3}}{3(4)} = \frac{-1}{36}$

$\therefore y - 4 = \frac{-1}{36}(x - \frac{1}{3}) \Rightarrow y - \frac{4}{} = -\frac{1}{36}x + \frac{1}{108} \Rightarrow y = \frac{-1}{36}x + \frac{433}{108}$ ($\frac{432}{108}$, $+\frac{432}{108}$)

on test b) & $\therefore$ $\boxed{\text{y-int } (0, \frac{433}{108})}$

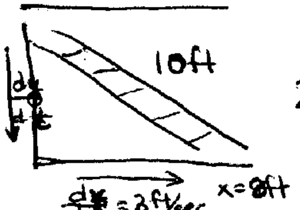
#7a) & #7b)

on Test a	On Test b
a) $v(t) = h'(t) = 128 - 32t$ $\therefore v(1) = 96$ ft/sec	a) $v(t) = 128 - 32t$ $\therefore v(7) = -96$ ft/sec
b) $a(t) = h''(t) = -32$ ft/sec² from a)	b) same as test a)
c) ave velocity $= \frac{h(3)-h(1)}{3-1} = \frac{240-112}{2} = 64$ ft/sec	c) ave velocity $= \frac{h(7)-h(5)}{7-5} = \frac{112-240}{2} = -64$ ft/sec
d) max height is found $h'(t) = 0$	d) same as test a)

$v(t) = 0 \Rightarrow 128 - 32t = 0 \Rightarrow 32t = 128 \Rightarrow \boxed{t = 4\text{ sec}}$

#8a & #8b

on Test a



$x^2 + y^2 = 10^2$

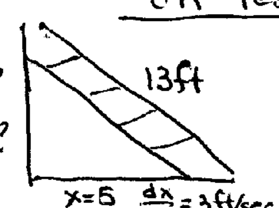
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{x\frac{dx}{dt}}{y}$

& $y = \sqrt{100 - x^2}$ so $= \sqrt{100-64} = \sqrt{36} = 6$ft

$\therefore \frac{dy}{dt} = \frac{-2(8)(3)}{2(6)} = \boxed{-4\text{ ft/sec}}$

on Test b



$x^2 + y^2 = 13^2$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = \frac{-x\frac{dx}{dt}}{y}$

$y = \sqrt{169 - x^2} = \sqrt{169-25} = \sqrt{144} = 12$ft

$\therefore \frac{dy}{dt} = \frac{-(5)(3)}{12} = \boxed{\frac{-5}{4}\frac{\text{ft}}{\text{sec}}}$

or -1.25 ft/sec

#9a & #9b

on Test #a	on Test #b
a) $f'(x) = 2x^3 - 18x$	a) Same as test a
$f''(x) = 6x^2 - 18$	

b) $f'(x) = 0 \Rightarrow 2x(x^2 - 9) = 0 \Rightarrow 2x(x+3)(x-3) = 0$ work for both tests

$x = 0 \quad x = \pm 3$

Key for Midterm #3a & 3b Con'd

on test a | **on test b**

#9a & 9b

on test a: b) So critical pt. in $[-4,1]$ are $x=0,-3$

on test b: b) So critical pt. in $[-1,4]$ are $x=0,3$

c&d) $f''(0)=-18<0 \therefore$ concave down & zero is a max
$f''(-3)=36>0 \therefore$ concave up & -3 is a min
$f''(3)=36>0 \therefore$ concave up & 3 is a min
} 2nd derivative test

or
$f'(-4)<0$ & $f'(-2)>0 \therefore \searrow \nearrow$ ($x=-3$) telling us -3 is a min
$f'(-2)>0$ & $f'(1)<0 \therefore \nearrow \searrow$ ($x=0$) telling us 0 is a max
$f'(1)<0$ & $f'(4)>0 \therefore \searrow \nearrow$ ($x=3$) telling us 3 is a min
} 1st derivative test

e) $f''(x)=0 \Rightarrow 6x^2-18=0 \Rightarrow 6(x^2-3)=0 \Rightarrow x=\pm\sqrt{3}$

on test a: $x=-\sqrt{3}$ is an inflection pt on $[-4,1]$ $\boxed{x\approx -1.7}$

on test b: $x=+\sqrt{3}$ is an IP on $[-1,4]$ $\boxed{x\approx 1.7}$

on test a:

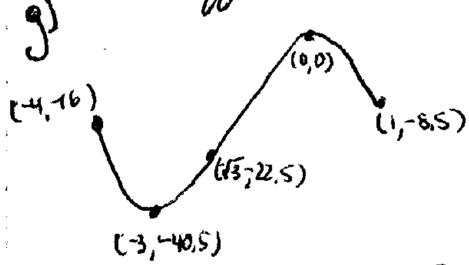
f) $f(-4)=\frac{1}{2}(-4)^4-9(-4)^2=-16$
$f(-3)=\frac{1}{2}(-3)^4-9(-3)^2=-40.5$
$f(-\sqrt{3})=\frac{1}{2}(-\sqrt{3})^4-9(-\sqrt{3})^2=-22.5$
$f(0)=\frac{1}{2}(0)^4-9(0)^2=0$
$f(1)=\frac{1}{2}(1)^4-9(1)^2=-8.5$

on test b:

f) $f(-1)=\frac{1}{2}(-1)^4-9(-1)^2=-8.5$
$f(0)=\frac{1}{2}(0)^4-9(0)^2=0$
$f(\sqrt{3})=\frac{1}{2}(\sqrt{3})^4-9(\sqrt{3})=-22.5$
$f(3)=\frac{1}{2}(3)^4-9(3)^2=-40.5$
$f(4)=\frac{1}{2}(4)^4-9(4)^2=-16$

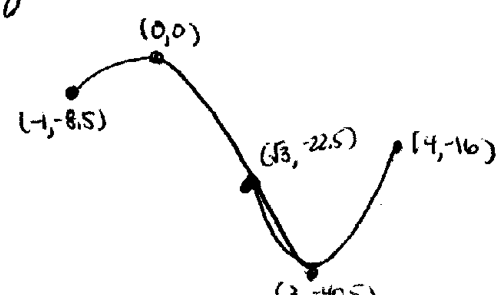
Don't use -1.7 or 1.7 to find value of function at I.P. because round-off error occurs!!

on test a: g)



on scaled axes!!

on test b: g)



#10a & #10b

on test a:

$F'(x)=f'(g(x))\cdot g'(x)$
$\therefore F'(3)=f'(5)\cdot g'(3)$ ($5 = g(3)$)
$=4\cdot 3=12$
$\boxed{\text{(ii) } 12}$

on test b:

$\therefore F'(0)=f'(2)\cdot g'(0)$ ($2 = g(0)$)
$=4\cdot 5=20$
$\boxed{\text{(iv) } 20}$

Key For Midterm #3a & #3b Con'd

#1a & #1b

$\lim_{x\to\infty} e^{-x^3} = \lim_{t\to-\infty} e^{t} = 0$

since as $x\to\infty$ $-x^3\to-\infty$ let $t=-x^3$

& $\lim_{x\to-\infty} e^{-x^3} = \lim_{t\to\infty} e^{t} = \infty$

since as $x\to-\infty$ $-x^3\to\infty$ let $t=-x^3$

$\boxed{\text{so } f(x) \text{ has a horizontal asymptote at } y=0 \text{ since as } x\to\infty \text{ the function approaches zero}}$

$\lim_{x\to\infty} e^{x^5} = \lim_{t\to\infty} e^{t} = \infty$

since as $x\to\infty$ $x^5\to\infty$ let $t=x^5$

so no asymptote as $x\to\infty$

&

$\lim_{x\to-\infty} e^{x^5} = \lim_{t\to-\infty} e^{t} = 0$

since as $x\to-\infty$ $x^5\to-\infty$ let $x^5=t$

$\boxed{\text{so } f(x) \text{ has a horizontal asymptote at } y=0 \text{ since as } x\to-\infty \text{ the function approaches zero}}$