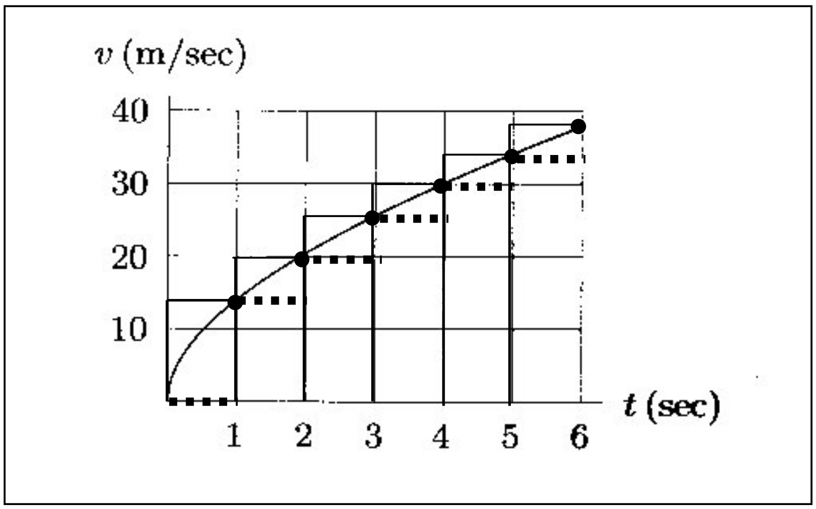
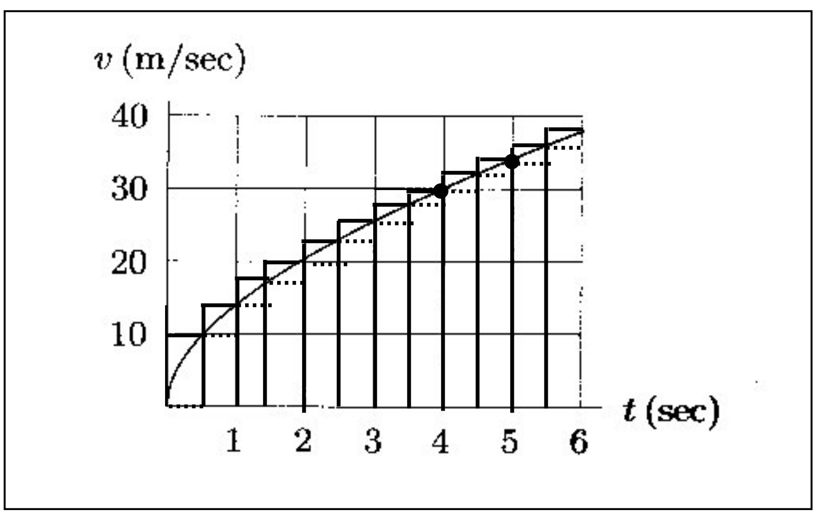
§5.2 The Definite Integral

As we saw in the last section – the small our bars, the less difference between the under and over estimates. So, in theory, if we were to take that bar width (i.e. the independent increments) to zero we'd be at an exact computation of the area under the curve.

Example: Recall the velocity, v, of an object (in m/s) shown by the graph below. In the last section we estimated the distance traveled on the interval t = 0 to t = 6 by using 2 unit wide bars. Let's recalculate the total distance using 1 unit wide bars and then 0.5 unit wide bars. *(#6 p. 238)





***Note**: It might be a good idea to make a table to help keep track of the information.

What we saw in this last example was that with each successive "chopping" of increments (increasing # of increments) that the approximation for the left hand sum is getting closer to the right hand sum and the area is getting closer and closer to the actual area under the curve.

Let's take care of some notation before we get too involved in a discussion:

t : The values of the independent

f(t): The values of the dependent; the function describing the curve

a to b: The interval under discussion for the independent

n: The number of bars (equal subdivisions of the interval a to b)

Δt: Width of each bar (change in the independent)

Notation used in our last example:

t : time (in seconds)

f(t): velocity (in m/s)

a to b: t = 0 to 6

n: 1^{st} w/ bars = 1 unit: 6 2^{nd} w/ bars = 0.5 unit: 12

Δt: 1^{st}: 1 second 2^{nd}: $^1/_2$ second

Recall:

Sigma Notation

Sum of values from 1 to some number, n

$$\sum_{i=1}^{n} x_i$$

Example: Find the average of 5 test scores, x's

$$\sum_{i=1}^{5} x_i \div 5$$

Sigma Notation used in our last example:

Left-Side Sums (under-estimates; touches on left corner of bar) $\sum_{i=0}^{n-1} f(t_i)\Delta t$

Right-Side Sums (over-estimates; touches on right corner of bar) $\sum_{i=1}^{n} f(t_i)\Delta t$

Note: *When the function is increasing the left side is the under estimate and the right side is the over estimate, but if the function is decreasing then the opposite is true. The left side is the over estimate and the right side is the under estimate.*

For n = 6 ➔ Left-side was

f(0) = 0, f(1) = 14, f(2) = 20, f(3) = 25, f(4) = 30, f(5) = 34

and 0 is the 1st independent value on the interval
1 is the 2nd " "
2 is the 3rd " "
3 is the 4th " "
4 is the 5th " "
5 is the 6th (and last) "

Note: *We don't use the nth (the 6th) in our calculation on the left! This is why we only go thru n–1. Like wise we start our sum process at the smallest value on this interval which is the 0 point.*

For n = 6 ➔ Right-side was

f(1) = 14, f(2) = 20, f(3) = 25, f(4) = 30, f(5) = 34, f(6) = 38

and 1 is the 1st independent value on the interval
2 is the 2nd " "
3 is the 3rd " "
4 is the 4th " "
5 is the 5th " "
6 is the 6th (and last) "

Note: *This time we are focused on the right side of the interval and we don't use the 0th and we do use the nth*

What we are trying to do is find out what the sum of all the rectangles are over an interval, a to b. We approach the value from the left and right trying to "hone in on it." Furthermore as we make our <u>increments smaller and smaller</u> which <u>increases the number, n</u>, increment we will get a value that eventually reaches a "true area." this is theoretical math and this is how we define a <u>**definite integral**</u>. Theory says that whether we "hone in on it" from the left or the right, we will eventually arrive at the area under the curve. That is to say, that as n ➔∞ (there are infinitely many subdivisions), the area computed as the sum of the over or under estimates will be the area under the curve from a to b. This is how we link the **Riemann Sum and a Definite Integral**.

$$\int_a^b f(t)\,dt$$

Notation:

$\int$: This stands for integral (comes from "S" for sum)

a ≤ t ≤ b: Limits of integration

$\int_a^b$: Endpoints of the integral

f(t): Function under which we wish to find the area

dt: Change in t (same as width of the bars; Δt)

When we have a function we will use our calculator to find a definite integral and later we will learn how to manipulate the functions.

<u>How To Use a TI to Find an Integral</u>

1) Use Y= to plot the function
2) Set the window so that you can see the limits of integration
3) Use 2nd Function Trace #7 to find the $\int f(x)dx$
   a) Set the lower limit = a
   b) Set the upper limit = b
   c) Enter
   d) Thinking & Returns $\int f(x)dx = \#$

Example: Estimate then calculate $\displaystyle\int_0^1 3^t\,dt$

Note: The estimation comes from what we were doing in §5.1 and earlier in this section. Just draw one big bar from 0 to 1 under the curve and use that single rectangle to approximate the integral (area under the curve).

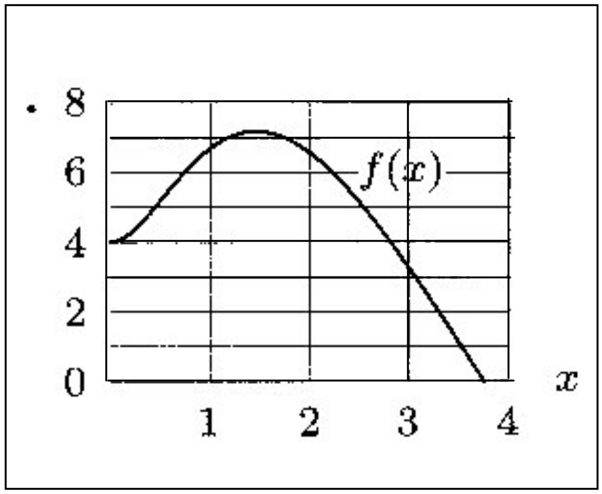
From a table or graph we have to continue to rely on our knowledge of the Reimann Sums (or for a graph use the grid squares for the area).

Example: Approimate the area under the curve for the function W(t) on the interval $3 \le t \le 4$. Use the correct integral notation to write the answer. *(#4 p. 246)

t	3	3.2	3.4	3.6	3.8	4.0
W(t)	25	23	20	15	9	2

***Note:** In my opinion, it really helps to draw a diagram to help see the bars. Notice decreasing so left & right are reversed in their size [right is smaller (bars that touch on rt. corner) than left (bars touch on lt. corner) sums]*

Example: Use the method from §5.1 & the method of squares to approximate the area under the curve shown in the graph below. *(#10 p. 246)



Pay Attention to Example #5 on p. 245
It is the number sense that connects Reimann Sums & the graph. The "just" is that left and right sums (1 big interval only) over and underestimates and the average is still too big so use it to gauge the approximate answer. This is what we used in problem #14 p. 246 to estimate.

§5.3 the Definite Integral as Area

As discussed in the last section, the area under a curve can be described by the sum of rectangles under the curve that over or underestimate the area. These were the Riemann Sums; either left or right-hand (so named by the corner that is touching the curve; over or under estimates). A **definite integral** comes from the theoretical discussion of taking the width of those rectangles to zero.

Area Under the Curve from a to b

for f(x) > 0 & a < b

$$\int_a^b f(x)\, dx$$

Recall from §5.2 that we can use our calculator to do examples like the following:

Example: Approximate the area under the curve using your calculator

$$\int_0^5 x^2\, dx$$

However, this is when the function is positive – when it is above the x-axis [f(x) > 0]. When the function is below the x-axis, the area is the opposite of the value you might expect. This leads to the fact that we must break a function that has both positive and negative values into parts where it is positive (above the x-axis) and negative (below the x-axis).

Area Under the Curve from a to b

for f(x) > 0 & f(x) < 0 for $a \le x \le b$

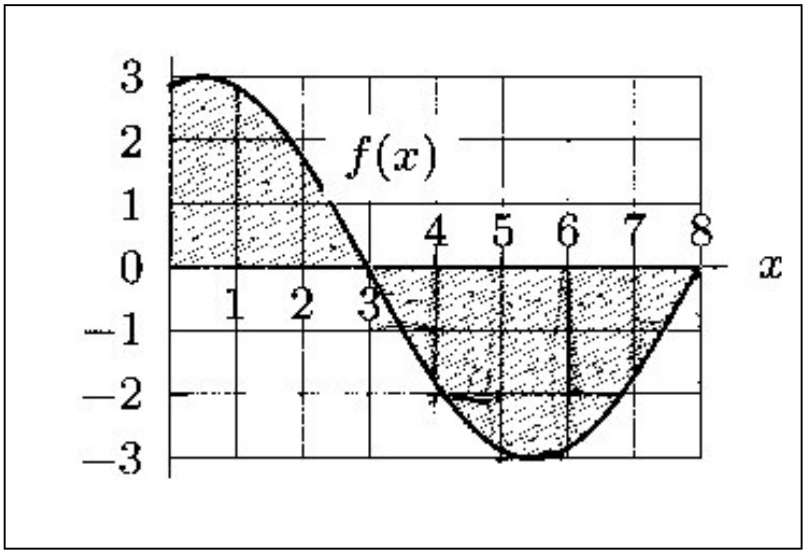
$$\int_a^b f(x)\, dx = \int_a f(x)\, dx + - \int^b f(x)\, dx$$

Note: *Break is where f(x) = 0.*

Note2: *Exercises 6-9 are checking to see if you get the picture that area above the curve is positive and area below the curve is negative.*

Let's use a graphic and a function example to exhibit this property of integrals.

Example: Use the graph to find the area under the curve between 0 and 8 and then use correct integral notation to represent the area.*(#10 p. 251)



Note: Use the number of rectangles to approximate the area. Be sure to count those above as positive and those below as negative.

Example: Use your calculator to find the area in 2 parts – the portion above the x-axis and the portion below the x-axis.

$$\int_0^2 x^3 - 3x^2 - 9x + 15 \, dx$$

Note: *You can find where the function dips below the x-axis by using the trace function on your calculator or by using good old fashioned algebra to solve for the x-intercepts – f(x) = 0.*

Next, we investigate the area between two curves. This we will need for application. (Think about cost and revenue functions.)

If g(x) & f(x) are 2 functions then the area between them can still be dvided into rectangles and the areas summed.

Area between g(x) & f(x)

if $g(x) < f(x)$ on $a \le x \le b$

$$\int_a^b f(x) - g(x)\ dx$$

Note: *a to b is usually going to be point between which f(x) = g(x).*

<u>Strategy for Finding Area Between Curves</u>

1) Visualize – Graph the two functions on your calculator
2) Find the points of intersection – Use algebra or your calculator's intersection feature.
   a) 2^{nd} Function Trace & Intersection (#5)
   b) Trace to just below an intersection and enter.
   c) Trace to just above an intersection and enter.
   d) Enter again & it will give coordinates of a guess
   e) Enter again & it will give the actual intersetion
   f) Repeat a-e for the second intersection

   Note*: If you don't trace to just below & just above intersection it may return an error*
3) Use graph function to graph $f(x) - g(x)$
4) Find the integral on the limits found in Step 2

Example: Find the area enclosed by $y = 3x$ & $y = x^2$

<u>§5.4 Interpretations of the Definite Integral</u>

Units of the definite integral is our first topic of discussion.

For:

$$\int_a^b f(x)\,dx$$

Recall: f(x) is the derivative (rate of change) of an original function thus its ***units*** are a ***quotient***.

dx is the change in *x* thus ***units*** are of the ***independent***

Hence: The units of $\int_a^b f(x)\,dx$ are units of the original functions' dependent variable.

This means that the definite integral's ***units*** are:

$$\boxed{\text{f(x)'s units multiplied by } x\text{'s units}}$$

To demonstrate units let's go over an example using each of our methods of interpretation: tables, graphic and formulas.

Example: Using the table below find an estimate for the coal produced between 1960 and 1990 if t is the number of years since 1960 and the rate is $1\text{x}10^{15}$ BTU per year. *#2 p. 256

Year	60	65	70	75	80	85	90
Rate	10.82	13.06	14.61	14.99	18.60	19.33	22.46

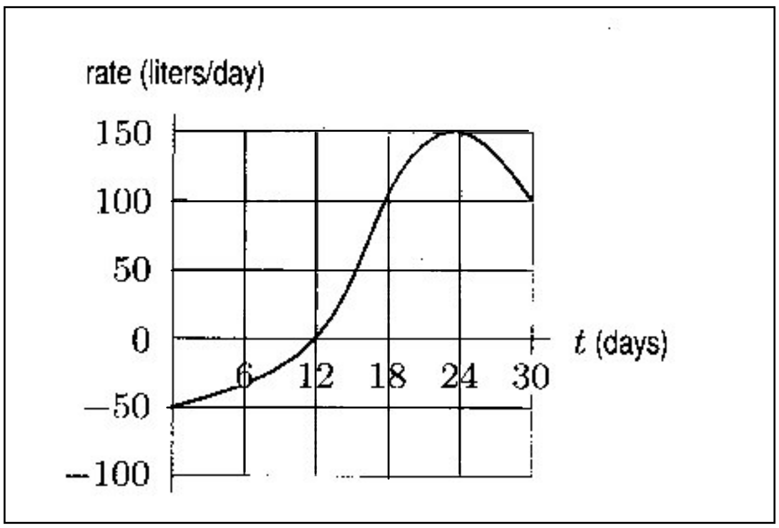
Note: *I really recommend a graphic of the data so you can see your left and right sums. This is the exact same method we were using in §5.1&5.2; recall exercises 2 p. 238 and 4 p. 246 that we completed as in class examples.*

*Applied Calculus, Hughes-Hallet et al, Ed. 4

Example: A 90ºC cup of coffee is placed in a 20ºC room when t = 0 and rate of change is $r(t) = -7(0.9)^t$ $^{\circ C}/_{min.}$ when t is in minutes. Estimate the coffee's temperature at t = 10. *(#8 p. 256)

Note: *This is what we did in §5.3 in exercise #20 on p. 247. You may also be wondering what this has to do with units – well, it is the unit interpretation of getting out ºC if you put in* $^{\circ C}/_{min}$ *• min., that allows us to get to the problem at hand.*

Example: For the graph below that shows the rate of change in the municipal water supply in L/day by time in days for the month of April. If the water supply on April 1 was 12,000 L, estimate the water supply on April 30.* (#14 p. 257)



Note: *This is like exercise #10 p. 251 where we counted the boxes to find the approximate area under the curve. Pay special attention to the fact that each box ≠ 1 L in this example, but instead is 50 x 6 = 300 L.*

*Applied Calculus, Hughes-Hallet et al, Ed. 4

Now a comparison of the two curves and what information we can learn from the differences in curves.

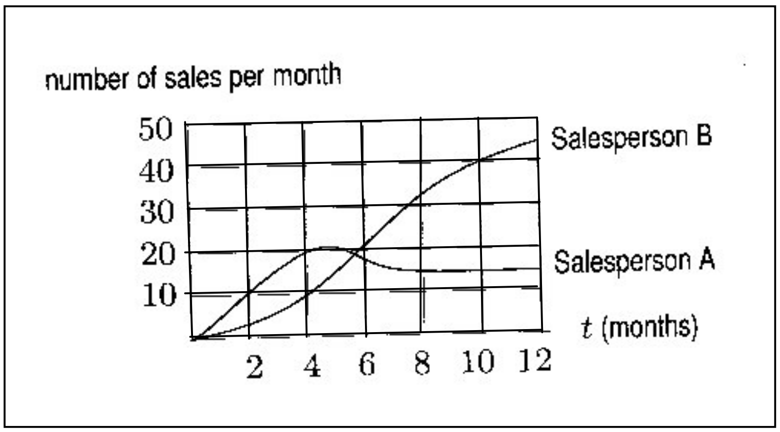
Example: Answer the following questions based on the graph of the #of sales per month by time in months for two salesmen A & B. *(#22 p. 258)

a) Which salesman has more sales after 6 months?

b) After 1 year which salesman has more sales?

c) At what time are the salemen's sales approximately equal?

d) What is the total sales at the end of the year for salesman A? B?



Note: *You are simply comparing the number of boxes again.*