<u>Introduction to Chapter 4</u>

Before we begin this chapter let me give you a preview. Our goal is to understand the ups and downs and general interesting points of a function. On one hand these interesting points can help us to visualize a function – graph the function. On the other hand, we have what the interesting features represent in the real world – the application. It is with this dual purpose that we will approach this chapter. In the first 2 sections we are trying to visualize and then quantify what interesting features we might need to know about a function. In the 3rd and 4th sections we will be applying our new knowledge and techniques to find define real world happenings.

In order to bring this into perspective let's think about a function that we give cursory attention to in Algebra but through our study in Calculus we can begin to have a better grasp of:

Example: $y = x^3 - x^2 - 8x + 2$

Recall: In Algebra you may have learned the general shape of this graph, but in graphing these you were normally given $f(x) = ax^3 + c$. Why? Because there are features of a cubic function that we can't attack easily with Algebra alone. You should know from Algebra that this function has an "S" shape, and the **points that are tough** are those where there are <u>peaks</u> and <u>valleys</u> and the <u>point in between</u>.

Calculus: You know how to find peaks and valleys already. Remember that a peak is where the curve is concave down and has an instantaneous slope of zero. A valley is where the curve is concave up and has an instantaneous slope of zero. Between a peak and a valley the concavity **must** change. Do you remember how to find those features using Calculus? I hope you recall that finding an <u>instantaneous slope</u> is finding the <u>first derivative</u>. I hope you recall that finding **concavity** is finding the **second derivative**. The rest is applying Algebra to the derivatives and the original formula.

So here is my recommendation for beginning this chapter:

1) Know what functions give what shapes, because if you have a general idea of what the function looks like from the start it will be easier to figure out if you have found out all you can using Calculus.

   Start w/ the graphs of these functions:

   Quadratic, Cubic, Square Root, Reciprocal, Absolute Value, Exponential (base e) & Logarithmic

2) Know how to find the zeros of a function using algebra. Recall that zeros are x-intercepts.

3) Know how to find an ordered pair (the position on a function) given the position of the independent. Realize that in Calculus the

independent's value doesn't change whether it is the original function, the first or the second derivative.

4) Know what you already know from Calculus

- First Derivative gives slope of an equation at a point (instantaneous slope)
- When the first derivative is zero we have a "flat" spot on the curve.
- The first derivative function will cross the x-axis (have a zero) when the original function has a "flat" spot (eg the slope is zero)
    - The second derivative indicates when slopes go from increasing to decreasing, in other words it tells us about concavity. This could help with peaks and valleys or with change from concave up to concave down or vice versa.
    - When the second derivative is zero the slopes are not changing, this is a point where concavity could have changed
    - When the 2nd derivative is positive a function is concave up. When the 2nd derivative is negative the curve is concave down.

Return to the Ex: You should be able to locate the peaks and valleys and in betweens of:

$$y = x^3 - x^2 - 8x + 2$$

1) The are zero that are concave up or down. Putting our knowledge together we can determine we need to find the zeros of the 1st derivative and then locate those on the original function.

$dy/dx = 3x^2 - 2x - 8$

$3x^2 - 2x - 8 = 0$ when $(3x + 4)(x - 2) = 0$

so at $x = {-}^4/_3$ and 2

Are they peaks or valleys or neither?

$d^2y/dx^2 = 6x - 2$

at $x = {-}^4/_3$ $\quad 6({-}^4/_3) - 2 = -8 - 2 = -10$

thus at $x = {-}^4/_3$ concave down

at $x = 2$ $\quad 6(2) - 2 = 12 - 2 = 10$

thus at $x = 2$ concave up

Where on the original function?

$f({-}^4/_3) = ({-}^4/_3)^3 - ({-}^4/_3)^2 - 8({-}^4/_3) + 2$

$= {-}^{64}/_{27} - {}^{16}/_9 + {}^{32}/_3 + 2 = {}^{(-64 + -48 + 288 + 54)}/_{27}$

$= {}^{230}/_{27}$

$({-}^4/_3, {}^{230}/_{27})$

$f(2) = (2)^3 - (2)^2 - 8(2) + 2 = 8 - 4 - 16 + 2 = -10$

$(2, -10)$

Now know there's a peak at $({-}^4/_3, {}^{230}/_{27})$ & valley at $(2, -10)$

2) Where's the middle where it goes from being a mountain to a valley? That's where the concavity changes.

$d^2y/dx^2 = 0$ so $6x - 2 = 0$

so $6x = 2$ $x = {}^1/_3$

$f({}^1/_3) = ({}^1/_3)^3 - ({}^1/_3)^2 - 8({}^1/_3) + 2$

$= {}^1/_{27} - {}^1/_9 - {}^8/_3 + 2 = {}^{(1 + -3 + -216 + 54)}/_{27}$

$= {}^{-64}/_{27}$

$({}^1/_3, -{}^{64}/_{27})$

From those 3 points and knowing the general shape you could make a fairly accurate approximation of the graph. Knowing from Algebra, the zeros of the functions itself and the y-intercept and it's symmetric point (symmetry of course being across the line y = x for this function) would improve the graph even further.

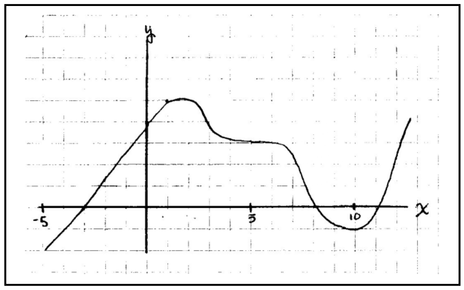
Now, with this all said, it is time to talk about the material in the sections of the chapter. You may want to return to this portion of the notes from time to time to remind yourself of what you do know!

§4.1 Local Maxima & Minima

A **critical point** is defined in Calculus where the first derivative is zero or undefined. Recall that this means the slope of the tangent line is zero.

Critical Point	$f'(x) = 0$ or Undefined

Example: Locate the critical points on the graph below.

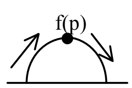


***Note:** When we talked about the points of interest in the introduction to the chapter we didn't talk about things that weren't peaks or valleys. They do exist as this example shows.

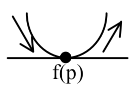
We can locate critical points by finding where the first derivative is zero and then checking out whether the derivative is posititve or negative to each side of the "flat" spot. This is known as the first derivative test.

First Derivative Test

Find where $f'(p) = 0$ & check values to the left and right



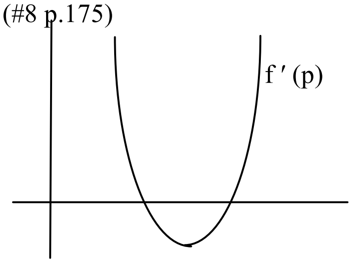
If $f'(\text{left}) > 0$ & $f'(\text{right}) < 0$ then p is a maximum



If $f'(\text{left}) < 0$ & $f'(\text{right}) < 0$ then p is a minimum

***Note:** If $f' > 0$ on both sides or if $f' < 0$ on both sides we have neither a maximum or a minimum. This will be our focus in the next section.

Example: Find the critical points and identify them as maximums or minimums.* (#8 p.175)



***Note:** This is testing your knowledge in reverse. When the derivative is zero the function is crossing the x-axis. When the function is above the x-axis it is a positive slope and when it is below the x-axis it is a negative slope.

*Applied Calculus, Hughes-Hallet et al, Ed. 4

Besides the first derivative we can use the second derivative to find where a maximum or minimum is located because maximums and minimums are located at points where the graph is concave up or concave down. So rather than see what the derivative looks like on either side, we could just see what the second derivative looks like at the point.

Second Derivative Test

If $f'' > 0$ then the curve is concave up and point is a Minimum

If $f'' < 0$ then the curve is concave down and point is a Maximum

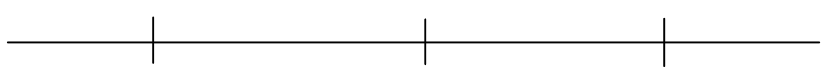
We will use the first and second derivative tests to investigate functions to find which critical points are maximums and which are minimums and which are neither.

Example: For the function $f(x) = 3x^5 - 5x^3$ answer the questions:
*(#12 p. 175)

a) Find the derivative.

b) Find the critical points of the function.

c) Using the regions created by the critical points, find where the first derivative is greater and less than zero. Place that information on the diagram below to assist in picturing of the function's maximum and minimums and which critical points are neither.



d) Find the second derivative.

f) Find the value of the second derivative at the critical points in b). What does this tell you about each critical point? Compare this to what you learned from the 1st derivatives in c).

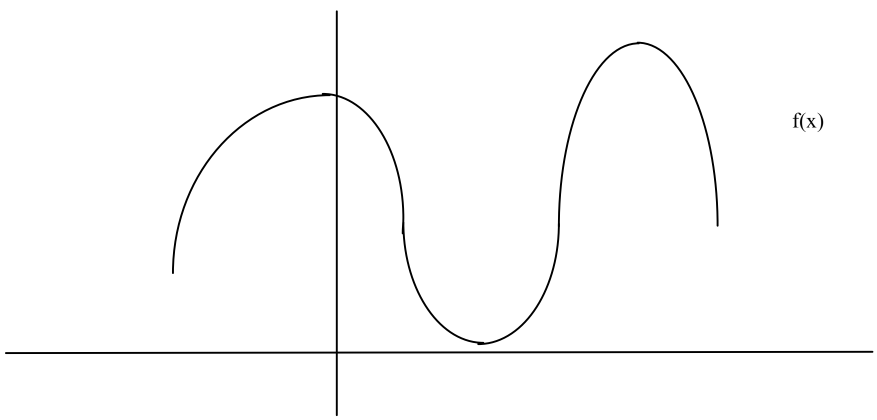
§4.2 Inflections

An **inflection point** is a point where the concavity of a curve changes. Recall that we know that concavity changes when the second derivative changes sign and at the instantaneous point that the sign changes, the second derivative must be zero.

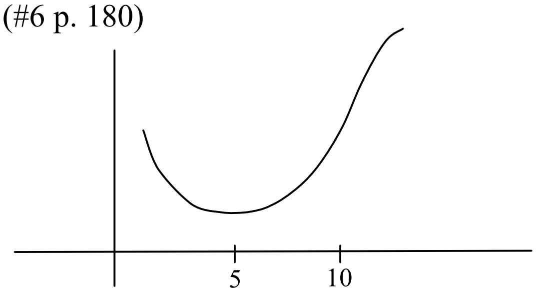
Inflection Point	$f''(p) = 0$ or Undefined

***Note**: Just as in Algebra in naming the intercept vs. the intercept point, there is also a difference in the point *p* where the inflection occurs and the inflection point's position on the graph of the function and you will know which by context. Not every place $f''(p) = 0$ or Undefined is an inflection point.

Example: Indicate the number number and position of the inflection points.
* (#4 p. 179)

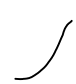


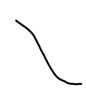
Example: Locate the critical point and the inflection point.
*(#6 p. 180)

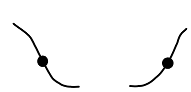


Now that we have become more adept at locating max, min and now inflection points, the reasonable question would be, "Why do we want to know?"

Concavity's Meaning

1) Concave up to Concave down ➔ Increase Growth to Decrease Growth 

2) Concave down to Concave up ➔ Decrease Growth to Increase Growth 

3) Concavity is zero ➔ Point of maximum growth 

We'll look at 2 problems to see applications of inflection points.

Example: During a flood, the water level in a river first rose faster and faster, then more slowly until it reached its highest point, then went back down to its pre-flood level. Consider water depth as a function of time. *(#8 p. 180)

a) Draw a graph of the water depth as a function of time.

b) Is the time of highest water level a critical point or an inflection point?

c) Is the time when the water first began to rise more slowly a critical point or an inflection point?

***Note:** From this problem we can see critical values and inflection points and how they come about in the real world. The inflection point represents the time where the water had reached its max increase and the maximum represents the highest water level.

The next example refers to an idea about limiting. For an exponential function such as e^{-t} we would like to investigate what happens when t goes infinity. If you investigate the graph of e^{-t}, you will see that as t gets larger and larger, the function value gets closer and closer to zero. Let's investigate by substituting and then you should also look at the graph on your calculator.

$$f(0) = e^{-(0)} = 1 \ \& \ f(1) = e^{-(1)} \approx 0.3679 \ \&$$
$$f(10) = e^{-(10)} \approx 4.54\text{x}10^{-5} \ \& \ f(100) = e^{-(100)} \approx 3.72\text{x}10^{-44}$$

And as you can see, it is getting very close to zero very quickly. Now, as we do the next problem, we need to remember that as time goes on, the exponential portion of the function will become zero so the value of our function will simply be the portion of the function that does not have to do with the independent variable. It also turns out that the inflection point has something to do with the "limiting value". The inflection point will occur at approximately 1/2 the limiting value.

Example: Captain James Cook introduced 10 rabbits to an island in 1774. The rabbit population on the island is approximated by: *(#27 p. 180)

$$P(t) = \frac{2000}{1 + e^{5.3 - 0.4t}}$$

a) Graph P using your calculator. It will be helpful to reset the viewing window for x's by 10 from 0 to 100 and y's by 10 from 0 to 2010.

b) What do you notice about the top of the graph? Use your trace function to find the y values (population size) at the top of the graph (as time is going to infinity)? Does what you are seeing agree with what was said about the limiting value?

c) Estimate the point at which the rabbit population was growing most rapidly by tracing along the graph where you see the "flat spot" and looking at the derivatives to see where the derivatives change from being bigger and bigger to smaller. Once you've found that area, zoom in on the area using your window key to more accurately estimate the value. Is the value you found approximately $^1/_2$ the value found in part b)?

d) Hypothesize about what could have caused the rabbit population to look like this.

Now that we've seen the visual approach, let's look at the inflection point from a strictly mathematical sense.

Example: Find the inflection point of $f(x) = x^3 - 18x^2 - 10x + 6$ algebraically

***Note:** This just means to find the second derivative and set it equal to zero. Be sure to check to see if you really have an inflection point. That means look at the value just to the left and the right to see if they change sign (concave up to down or vice versa). If you think about the graph of the function you will know to expect 1 inflection point. Give the point on the graph too, not just the value of the independent.

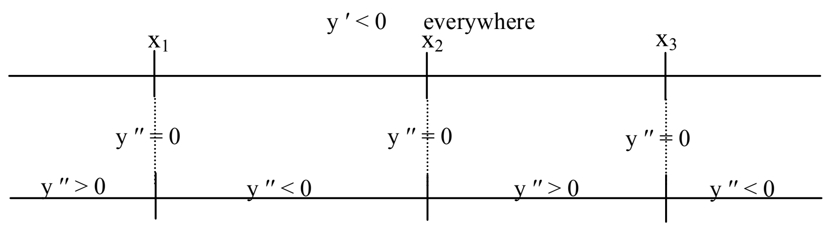
Example: Use the first derivative to find the critical points and the second derivative to find the inflection points. Identify them as ordered pairs. Using your knowledge of the maximums and minimums found using algebra, set your calculator's viewing window to see the graph and ID the critical values as local maximums, minimums or neither.*(#18 p. 180)

$$f(x) = x^4 - 4x^3 + 10$$

*Applied Calculus, Hughes-Hallet et al, Ed. 4

Just to make sure we have the concepts down, we'll look at the reverse scenario. We will start from the derivatives and try to understand the original function.

Example: Sketch a possible graph of the function y = f(x) based on the first and second derivatives. * (#24 p. 180)



There are multiple urn problems in this section. Where the urn has a drastic change in shape causes a change in fill rate, hence we see inflection points. When the shape of the urn is a smooth curve, the point of the largest diameter causes an inflection point.

Example: Look at the urn in problem #30 on p. 181. The top is a cylinder but the bottom has a square shape to it. Sketch a graph of the depth of the water per unit of time. Mark on the graph where the water reaches the corners of the vase.

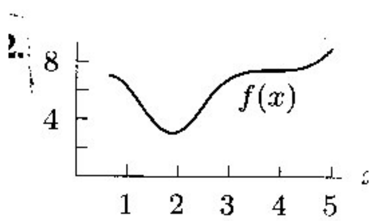
§4.3 Global Maxima & Minima

Optimization is the study of finding maximum and minimum values.

> **Global Minimum**
> If @ p, f(p) is less than all other values then p is a minimum
>
> **Global Maximum**
> If @ p, f(p) is greater than all other values then p is a maximum

*Don't forget that our second derivative can indicate whether we have a maximum or minimum since it tells us if a function is concave up or down.

Example: Locate all the maximum and minimum and indicate whether they are local or global. Remember that globals can be both.



How to Find Local/Global Min&Max

1) Find the critical points (remember set the first derivative equal to zero and solve & check concavity)
2) Sketch the graph
3) Compare endpoints w/ critical points

Example: For $f(x) = x^3 - 3x^2$ on the interval $-1 \leq x \leq 3$ answer all the following questions. *(#18 p. 186)

a) Find the first derivative.

b) Find the second derivative.

c) Find the critical points.

d) Find the inflection points.

e) Evaluate the function at the critical points and endpoints

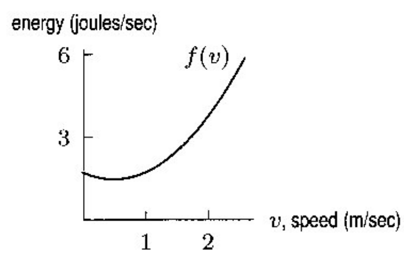
f) ID local and global max & min

Example: Find the global maximum/minimum for $f(x) = x - \ln x$ for $x > 0$ *(#26 p. 186)

<u>Optimizing f(x)/x</u>

Visually this is finding the tangent line through the origin with the smallest slope. To see this let's look at an example.

Example: The following graph shows, f(v), the amount of energy a bird consumes (by flying) in j/sec as a function of speed, v, in m/sec. The function a(v) is the amount of energy consumed in j/m [in other words it is f(v)/v]. To see where a(v) is minimize (optimized) we need to investigate the lines tangent to the curve through the origin.



<u>Optimization Problem Strategies</u>

1st realize that many times the function that we will need to optimize won't be given directly to us. We may need to find the function 1st.

Step 1: Construct the model that relates to the variable that you wish to optimize (it's the dependent)

Step 2: Find the critical values of the function

Step 3: Find the global max/min on the interval

Step 4: Use your model to answer your question(s)

*Applied Calculus, Hughes-Hallet et al, Ed. 4

Example: What value of w minimizes S if $S - 5qw = 3qw^2 - 6pq$
* (#30 p. 186)

1) Start by solving for S

2) Find the derivative of S with respect to w

3) Set $dS/dw = 0$

4) Solve for w

Example: The energy expended by a bird per day, E, dependson the time spent foraging for food per day, F hours. Foraging for a shorter time requires better territory, which then requires more energy for its defense. Find the foraging time that minimizes energy expenditure if: *(#32 p. 186)

$$E = 0.25\,F + 1.7F^{-2}$$

1) Find dE/dF

2) Set equal to zero and solve for F

Example: During the flu outbreak in a school of 763 children, the number of infected children, I, was expressed in terms of the number of susceptible (but still healthy) children, S, by the expression

$$I = 192 \ln \left({}^{S}/_{762}\right) - S + 763$$

What was the maximum possible number of infected children?*(#36 p. 186)

This is the same thing we did before, but because the point is a global maximum we are finding the maximum value instead of a minimum value.
Once we solve for S, we can substitute in and find the value for I to find the actual maximum value. Notice that the previous problems all said to find the value of the independent that would minimize the dependent, not to find the minimum dependent value.

§4.4 Profit, Cost & Revenue

Recall:

$\pi(q)$ is Profit
$\pi(q) = R(q) - C(q)$
$MC = C'(q)$
$MR = R'(q)$
$C' < R'$ means if $\pi > 0$ production can go ↑ (section 2.5)
$C' > R'$ means if $\pi > 0$ production can go ↓

From what we know

When $C' = R'$ production doesn't need to ↑ or ↓ and thus π is at its max/min

Maximize/Minimize Profit
$C' = R'$
or
$\pi' = 0$

Process for finding Max/Min Profit:

1) Find C' & R'
2) Set $C' = R'$ and solve for quantity, q
3) Is it a maximum or a minimum?
   a) Check to left for C' & R' values When $C' < R'$ (max) & $C' > R'$ (min)
   b) Check to right for C' & R' values When $C' > R'$ (max) & $C' < R'$ (min)
4) Global?
   a) Check values at the endpoints for $\pi = R - C$
   b) Compare a)'s values to values of π at 2)'s quantity

Maximizing Revenue

- Demand Equation: $q = \dfrac{\#}{\text{increase/decrease in p}} \cdot \text{price } (p) + \text{baseline}$
- Recall: $R = p \cdot q$ where q is the demand equation
- Set $R' = 0$ and solve for p
  This is the p to get max R

Example: An ice cream company finds that at a price of \$4 demand is 4000 units. For every \$.25 decrease in price, demand increases by 200 units. Find the price and quantity sold that maximizes revenue.
* (#22 p. 194)

Example: The following table gives the cost and revenue in dollars for different productions levels, q. Answer the questions that follow.
*(#20 p. 194)

Q	0	100	200	300	400	500
R(q)	0	500	1000	1500	2000	2500
C(q)	700	900	1000	1100	1300	1900

a) At approximately what production level is profit maximized?

b) What price is charged per unit for this product?

c) What are the fixed costs of production?