§3.1 Derivative Formulas for Powers and Polynomials

First, recall that a derivative is a function. We worked very hard in §2.2 to interpret the derivative of a function visually. We made the link, in Ex. 28 on p. 100 of our book that the derivative can be quantified by a formula. In this section, we begin the process of how to derive the formula for the derivative based upon some manipulation of the function's formula. We can look at the manipulation of the derivative as a manifestation of translation and algebra of functions that we discussed in Ch. 1 to try to make sense of why we do what we do. I will point this out, but I figure for the most part, many of you will simply memorize what to do.

A Constant Function

The constant function is a horizontal line. The slope of any horizontal line is zero and hence the derivative of the constant function, which is the slope of the tangent line, is therefore zero.

If $f(x) = k$, then $f'(x) = 0$

Example: Find the derivative of: $y = 5$ *(#1p.139)

A Linear Function

Any linear function has is a straight line, and the slope of tangent lines will all be the same – the slope of the line. Hence the derivative of a linear function is the slope of the function.

If $f(x) = b + mx$, then $f'(x) = m$

Example: Find the derivative of: $y = 5x + 13$ *(#8p.139)

The Power Rule

The power rule is a manipulation of a power function. Recall in §1.9 we learned how to identify a power function and rewrite it as $y = kx^p$. That skill will be necessary here as well.

If $f(x) = x^p$, then $f'(x) = p \bullet x^{p-1}$

Example: Find the derivative of: $y = x^{-12}$ *(#4p.139)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Derivative of A Constant Times a Function

This really is a shrinking/stretching translation of a function. As such, we know that the original function is only changed by a multiple and therefore the slope of the tangent lines are only changed by that multiple as well.

If $f(x) = c[g(x)]$, then $f'(x) = c[g'(x)]$

Example: Find the derivative of: $y = {}^{4}/_{3}\pi r^2 b$ with respect to r.
*(#34p.139)

Derivative of Sum/Difference

This is really the algebra of functions being put to use on the derivative. Since the derivative is a function the rules still apply.

If $h(x) = f(x) + g(x)$, then $h'(x) = f'(x) + g'(x)$
or
If $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x)$

Example: Find the derivative of: $v = at^2 + \frac{b}{t^2}$ with respect to t.
*(#32 p. 139)

Derivative of Polynomials

Polynomials are really just sums of power functions so this allows us to simply apply the derivative of a sum or difference.

Example: Find the derivative of: $y = 8t^3 - 4t^2 + 12t - 3$
*(#14 p. 139)

Now, are you ready for a challenge? What will you do with this one to apply what you know?

Example: Find the derivative of: $h(\theta) = \theta(\theta^{-1/2} - \theta^{-2})$ wrt θ
*(#28 p. 139)

Finding Rate of Change at a Point

If we wish to find the rate of change at a particular point, we want to find the slope of the tangent line. All tangent lines have a slope given by the derivative formula, so we can evaluate the derivative formula at an independent to find the rate of change at that point.

Example: Find the rate of change of a population of size $P(t) = t^3 + 4t + 1$ at time, $t = 2$. *(#40 p. 139)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Other ways to use the derivative formula?

To find the tangent line's equation, of course – recall from §2.3 and investigation using our calculator in §2.1 notes that we can find the equation of tangent lines at independent values.

1) We can use the derivative formula to find the slope of the tangent line at any point in the domain.
2) From there, we can use the original function to find an ordered pair that lies on the tangent line.
3) Finally, we can use the point-slope form to give the equation of the tangent line using the point and the slope.

Example: Find the equation of the tangent line of $f(x) = x^3$ at the point where $x = 2$. *(#50a, p. 140)

The Second Derivative Formula

The second derivative is the derivative of the derivative formula. It can be found using all the rules that we just discussed for the first derivative. Furthermore, the range of the second derivative formula tells us about the concavity of the original function.

If $f''(x) > 0$ on a portion of the domain,
then the function is *concave up* on that domain

If $f''(x) < 0$ on a portion of the domain,
then the function is *concave down* on that domain

Example: For the function: $f(t) = t^4 - 3t^2 + 5t$ *(#46 p. 139)

a) Find the first derivative.

b) Find the second derivative.

c) On what values of the domain is f(t) increasing?
(Hint: This requires looking at the graph and finding where range is zero, which will require your skills of finding the x-intercept of a parabola.)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Applications in Cost and Revenue
Recall from our study in §2.5 that the cost and revenue function's derivatives tell us about the cost to make or the revenue brought in (marginal cost/revenue) incurred/brought in by producing one additional unit beyond the given "n".

Example: The demand curve for a product is given by $q = 300 - 3p$, where p is the price of the product and q is the quantity that consumers buy at this price. *(#56 p.140)

a) Write the revenue as a function of the price, *p*. (Recall from Ch.1.2 that $R = p \bullet q$)

b) Find the marginal revenue at a price, *p* of $10 (in other words, $R'(q)$) and interpret your answer using units.

c) For what prices is the marginal revenue positive? ($R'(q) > 0$)

f) For what prices is the marginal revenue negative? ($R'(q) < 0$)

* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

§3.2 Exponential and Logarithmic Functions

The derivative of the exponential function can be seen to look like the function itself. The derivative function for all bases except the base "e", is proportional to the original function.

If $f(x) = a^x$, then $f'(x) = (\ln a)a^x$

*If $f(x) = e^x$, then $f'(x) = e^x$

*This follows from base "a" since ln e is equal to 1!

Example: Find $f'(x)$ for: $f(x) = x^3 + 3^x$
*(#4 p.144)

Example: Find $^{dy}/_{dx}$ for: $y = B + Ae^t$
*(#20 p. 144)

***Note:** don't forget in §3.1 we learned that the derivative of f(x) + g(x) is f′(x) + g′(x), and that the derivative of a constant is 0 and that the derivative of c•f(x) is c•f′(x).

If we add a constant to the function we find the derivative to be the constant times the derivative of the function.

If $f(x) = e^{kt}$, then $f'(x) = ke^{kt}$

***Note**: Since $(e^k)^t$ so $\ln e^k = k$ and thus k times e^{kt}. Stay tuned for the chain rule which will give us further insight into this derivative.

Example: Find the derivative of: $y = e^{0.7t}$
*(#12 p.144)

Earlier we saw an example that investigated the natural log function. We saw through investigation of the approximate derivative (based upon rate of change on a very small interval) that the derivative function appeared to be 1 over the argument. We see here that is the case.

If $f(x) = \ln x$, then $f'(x) = {}^1/_x$

Example: a) Find $^{dD}/_{dp}$ for: $D = 10 - \ln p$
*(#24 p. 144)

b) Find $R'(q)$ for: $q^2 - 2\ln q$
*(#26 p.144)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: For the function $f(t) = 4 - 2e^t$ answer the following questions. *(#29 p. 145)

a) Find $f'(t)$

b) Find $f'(-1)$, $f'(0)$, $f'(1)$

c) Find $f(-1)$, $f(0)$ and $f(1)$

d) Give the equations for the tangent lines at $x = -1, 0, 1$ (see §3.1)

e) Use your calculator to graph $f(t)$.
(You may want to use the Window menu to dial into the regions)

f) Using the graph in e), write down the derivatives given by your calculator and the tangent lines at $x = -1, 0, 1$

g) Now plot $f'(t)$ and compare the graphs of $f(t)$ and $f'(t)$. What do you notice about the shape? How does this compare to what we know about the derivative of $f(x) = e^x$

Example: At a time t hours after administered, the concentration of a drug in the body is $f(t) = 27e^{-0.14t}$ ng/ml. *(#42 p. 145)

a) What is the concentration 4 hours after the drug was administered?

b) At what rate is the concentration changing at that time?

Example: The cost of producing a quantity, q, of a product is given by:

$$C(q) = 1000 + 30e^{0.05q} \quad \text{dollars}$$

*(#43 p.145)

a) Find the cost at $q = 50$.

b) Find the marginal cost at $q = 50$. Interpret in terms of economics.
(Refer to §2.5 if you can't remember marginal cost.)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§3.3 The Chain Rule

If a function is a function of another function then take the derivative of the outer-most function and then multiply it by the derivative of the inner function.

If $y = f(z)$ and $z = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dz} \cdot \frac{dz}{dt}$

or

If $f(g(t))$, then $\frac{d}{dt} f(g(t)) = f'(g(t)) \cdot g'(t)$

***Note:** You can use substitution to solve like in Algebra when you did factoring of $(x^2 + 1) – 4(x^2 + 1) – 5$ or when you solved polynomial equation by using quadratic form, eg $x^{-2} + 12x^{-1} + 35 = 0$

Example: Find $^{dw}/_{dt}$ for: $w = (t^2 + 1)^{100}$
*(#4 p. 150)

***Note**: You can think of $z = t^2 + 1$ and look at the derivative of z^{100}, then switch off and get the derivative of $t^2 + 1$, that's what the chain rule does.

Example:

a) Find $^{dw}/_{dt}$ for: $w = e^{-3t^2}$
*(#12p. 150)

b) Find $^{dy}/_{dx}$ for: $y = 12 – 3x^2 + 2e^{3x}$
*(#14p.150)

c) Find $f'(x)$ for: $f(x) = 6e^{5x} + e^{-x^2}$
*(#10p. 150)

d) Find $f'(t)$ for: $f(t) = \ln(t^2 + 1)$
*(#16p. 150)

e) Find $^{dQ}/_{dt}$ for: $Q = 100(t^2 + 5)^{0.5}$
*(#22p. 150)

***Note:** Make sure you have firmly in mind whether you are dealing with an exponential or a power function.

f) Find $^{dy}/_{dx}$ for: $y = \sqrt{e^x + 1}$
*(#26p. 150)

g) Find $f'(\theta)$ for: $f(\theta) = (e^{\theta} + e^{-\theta})^{-1}$
*(#28p. 150)

Now to recall some things that we have discussed briefly from time to time and use them in some problems:

Physics

Velocity is the derivative of the function of position with respect to time. It is the change in position with respect to time. (§2.1 p. 88-89)
Acceleration is the derivative of the velocity. It is the rate of change in the velocity over time, $^{dv}/_{dt}$. (§2.3 p. 104)

Example: The distance, s of an object at some time, t can be described by:

$$s = 20e^{t/2}$$

Give the function that represents the velocity. *(#41 p. 151)

An interesting concept that we will see later is raised by problem #42 on p. 151 so we will investigate it now both for it's relation to Economics and for it's relation to material we will cover later in the quarter.

Example: B = \$ in an account at t years at an annual rate of r%

$$B = P(1 + r/100)^t$$

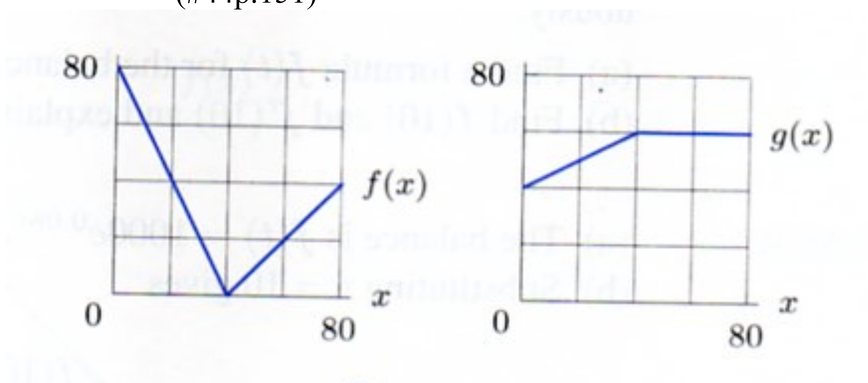
a) Find $^{dB}/_{dt}$ and explain what it means in terms of money.

b) Find $^{dB}/_{dr}$ and explain what it means in terms of money.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

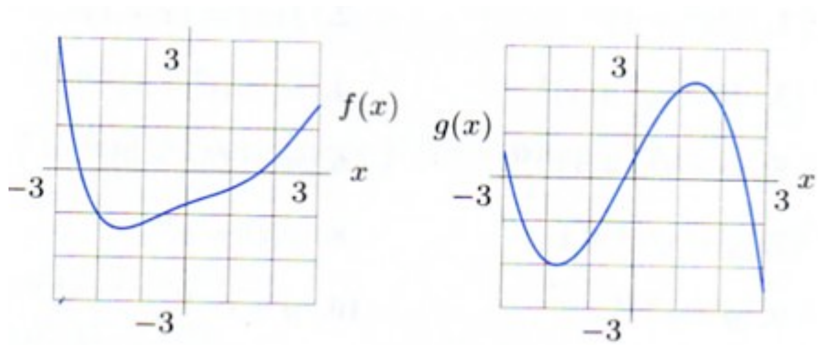
I would like to investigate 2 problems where we use a graph to interpret the chain rule.

Example: Using the graphs find: $\left.{}^{d}/_{dx}\, f(g(x))\right|_{x=30}$
*(#44p.151)



***Note:** Evaluate g(x) then find the derivative of that value for f(x) & find the derivative of g(x). Don't forget that derivatives are slopes!

Example: Using the graphs below, find h ′ (1) if h(x) = f(g(x))
*(#48p.151)



***Note:** We can only approximate as we don't know the functions' equations and we don't have fine detail.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§3.4 The Product and Quotient Rules

If we multiply two functions the derivative is the sum of each of the derivatives times the other function.

Product Rule:

If h(fg), then $h' = f'(t) \cdot g(t) + g'(t) \cdot f(t)$

or

$$\frac{d(uv)}{dx} = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$$

Example: Find $^{d}/_{dt}$ for: $f(t) = te^{-2t}$
*(#4p.154)

If we divide two functions the derivative is the difference of the numerator's derivative times the denominator function and the denominator's derivative times the numerator function, all divided by the denominator function squared.

Quotient Rule:

For $u = f(x)$ & $v = g(x)$

$$(f/g)' = \frac{f'(x) \cdot g(x) - g'(x) \cdot f(x)}{g^2}$$

or

$$\frac{d(u/v)}{dx} = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2}$$

Example: Find $^{dw}/_{dy}$ for: $w = \frac{3y + y^2}{5 + y}$ *(#27p.154)

Now, let's put this together with our rules from before.

Example: Find the derivative for:*(#6&18p.154)

a) $y = t^2(3t + 1)^3$

b) $f(t) = te^{5 - 2t}$

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.