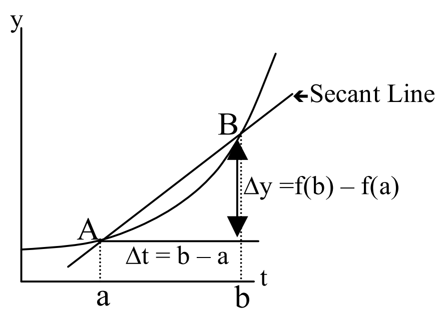
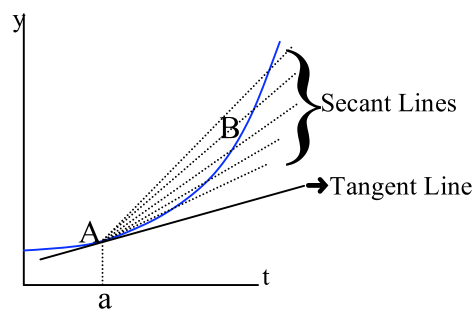
§2.1 Instantaneous Rate of Change

In this section we will revisit the idea of *average rate of change* from §1.3 and link it with the concept of ***instantaneous rate of change***. In section 1.3 we learned about the secant line and how it shows the slope of a function over time, which is what we call the average rate of change. We saw an example in the book about a grapefruit being thrown into the air and how the velocity, the change in distance over intervals of time changed. We also discussed an example of how a car travels on a freeway and as traffic speeds up and slows down the velocity changes, but overall the average velocity is slope of the tangent line. In this section we can liken the **instantaneous rate of change** to looking down at our speedometer at any point on the freeway trip and reading the velocity. Taking average rate of change over successively smaller increments of time (until the limit is reached, in other words the value where the rate of change becomes a constant) leads to the estimate of an instantaneous rate of change. We will visualize the instantaneous rate of change as the slope of a line called a **tangent line**.





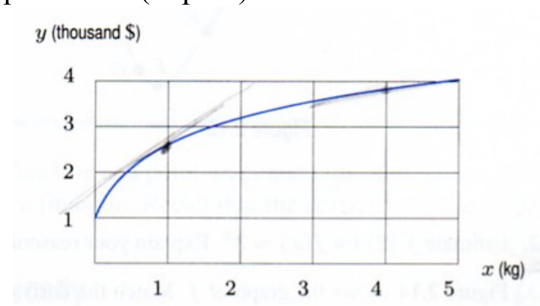
This process of finding the slope of the tangent line, is called finding the **derivative** of a function at some *a*. So the derivative of a function is the instantaneous rate of change at some value of the independent.

f ′ (a) is the derivative of f(x) at *a* and is 1) equivalent to the instantaneous rate of change 2) the slope of the tangent line 3) an average of the slopes of 2 secant lines equidistant from the point a 4) $\frac{f(a+h) - f(a)}{h}$ h = a very small increment

Now, we'll do examples that exhibit this concept in terms of graphs, tables, functions and descriptions of scenarios. Our goal is to make sure we can relate secant lines and tangent lines to slopes and to one another and differentiate between average and instantaneous rate of change and to use the slopes to indicate increasing and decreasing functions.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Use the graph below that shows the cost, y = f(x), of manufacturing x kilograms of a chemical, to answer the following questions.*(#6p.93)

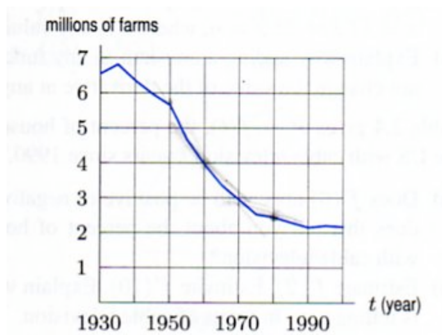


a) Is the average rate of change of the cost greater between x = 0 and 3 or between x = 3 and 5? Explain graphically.

b) Is the instantaneous rate of change of the cost of producing x kilograms greater at x = 1 or 4? Explain graphically.

c) What are the units of these rates of change?

Example: The graph below shows N = f(t), the number of farms in the US between 1930 and 2000 as a function of year, t.*(#2p.93)



a) Is f ′ (1950) positive or negative? What does this tell your about the number of farms?

b) Which is more negative f ′ (1960) or f ′ (1980)? Explain.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: A car's position, s, is given by the following table.*(#4p.93)

t (sec)	0	0.2	0.4	0.6	0.8	1.0
s (feet)	0	0.5	1.8	3.8	6.5	9.6

a) Find the average velocity over the interval $0 \leq t \leq 0.8$.

b) Estimate the velocity at $t = 0.2$

Note: *The average velocity is the slope of the secant line. The velocity at a point is instantaneous velocity, or the slope of the tangent line. A good estimate of the slope of the tangent line from a table that doesn't have small increments of the independent and dependent defined is to take the slope above and below the point and average. If both don't exist, the next best thing is to use the slope of closest secant line as an approximation.*

Example: Estimate $f'(2)$ for $f(x) = 3^x$. Explain your reasoning.*(#12p.94)

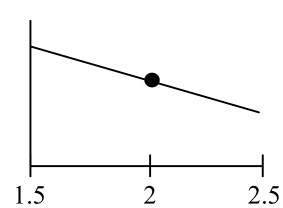
Note: *I'm going to take the position of problem #9 and do this over successively smaller intervals so that we can see the idea of limiting as well.*

Example: Graph the function $g(t) = 0.8^t$ on your calculator and then follow the instruction set below to answer the questions **a)** Determine whether $g'(2)$ is positive, negative or zero. **b)** Use a small interval to estimate $g'(2)$.* (#10p.94)

Step 1: $\boxed{\text{Y=}}$ 0.8 $\boxed{\wedge}$ t $\boxed{\text{X, T, }\theta\text{, n}}$ $\boxed{\text{ZOOM}}$ 6

Step 2: $\boxed{\text{TRACE}}$ to approximately x = 2

Step 3: $\boxed{\text{WINDOW}}$ x's 1.5 to 2.5 by 0.1 and y's 0 to 1 by 0.1 $\boxed{\text{GRAPH}}$



Note: *There are several ways to go from here:*

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

1) The most rudimentary is to manually compute a slope by TRACING up and down the function and getting (x,y) coordinates to either side of the point and using them to compute a slope.

2) A way to let our calculator find the derivative is to [TRACE] to x = 2 and then input the following.

[2nd] [TRACE] 6:dy/dx [↵] [↵]

This will give you the slope at x = 2.

3) Another way to let your calculator find the derivative is to find the tangent line and its slope. I again recommend the use of the WINDOW created above to zoom in on x = 2.

[2nd] [PRGM] 5:Tangent [↵] [↵]

This will display the tangent line and give its equation.

§2.2 The Derivative Function

This section takes the idea that for every value of x there is a value of the derivative and reminds you that that makes the derivative a function! If we find the value of the derivative for values of x it is then possible to graph the **derivative function**.

There are a few important things to keep in mind when shifting back and forth between the original function and the derivative function or vice versa.

Looking at a **graph** and f(x) ➔ f ′ (x):

1) Mark off on the graph the regions on which the slope is positive and negative. This will show where the derivative is positive (above the x-axis) and negative (below the x-axis). Keep in mind that these are the y values of f ′ (x) and that the x values are the same as those of f(x).
2) Anywhere that the f(x) has a maximum or minimum is a zero for f ′ (x). Recall that the zero of a function is where it crosses the x-axis. These places are f ′ (x)'s x-intercepts.

Looking at a **table** of values for f(x)

1) Use the $\frac{f(a+h)-f(a)}{h}$ idea to find the derivative's values
2) Again watch for positive and negative values of the derivative and zeros as you compute 1)

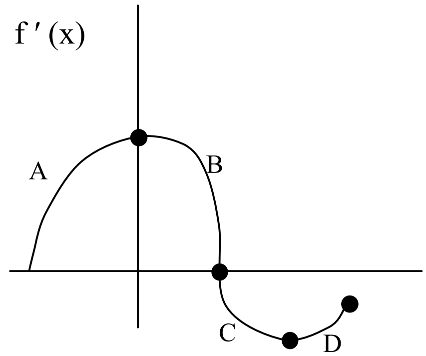
Working from f ′ (x) ➔ f(x)

1) When f ′ (x) is positive then the original function, f(x) was increasing. (y ↑ as x↑)
2) When f ′ (x) is negative then the original function, f(x) was decreasing. (y ↓ as x↑)
3) When f ′ (x) has an x-intercept then the original function, f(x) had a maximum or minimum
4) If f ′ (x) is decreasing, then the original function was concave down (slopes of tangent/secant lines getting < & <)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

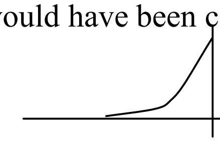
5) If $f'(x)$ is increasing, then the original function was concave up (slopes of tangent/secant lines getting > & >)

Example: Let's examine this graph of a derivative to see what the original f(x) may have looked like.

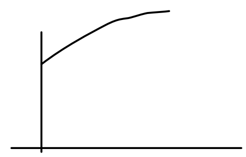


In this example:

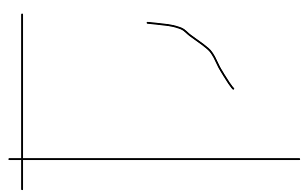
In region A, $f'(x)$ is increasing and positive ∴ the original function would have been concave up and increasing:



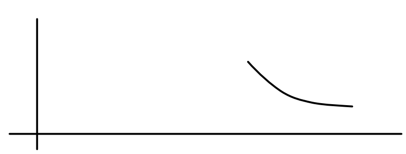
In region B, $f'(x)$ is decreasing and positive∴ the original function would have been concave down and increasing



In region C, $f'(x)$ is decreasing and negative ∴ the original function would have been concave down and decreasing. Note that there would have been a maximum because the derivative function crossed the x-axis:



In region D, $f'(x)$ is increasing and negative ∴ the original function would have been concave up and decreasing.



* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Draw the derivative functions together in the groups that you have been assigned.

a) $f(x) = ax$
b) $f(x) = ax^2$
c) $f(x) = ax^3$
d) $f(x) = a\sqrt{x}$
e) $f(x) = a\,|\,x\,|$

Example: Draw the graph of the continuous function, $y = f(x)$ that satisfies the following three conditions.*(#26p.100)

$f'(x) > 0$ for $1 < x < 3$
$f'(x) < 0$ for $x < 1$ and $x > 3$
$f'(x) = 0$ at $x = 1$ and $x = 3$

Example: Using the table of values approximate $f'(x)$ and indicate where the rate of change is positive, negative and approximately equal to 3. *(#20p.100)

x	2.7	3.2	3.7	4.2	4.7	5.2	5.7	6.2
f(x)	3.4	4.4	5.0	5.4	6.0	7.4	9.0	11.0
f ′(x)								

Example: If $f(x) = \ln x$, use small intervals to estimate the value of its derivative at 1, 2, 3, 4, and 5. Use the values to guess at a formula for the derivative of $f(x) = \ln x$. * (#28p.100)

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§2.3 Interpretations of the Derivative

This section will require the most from you in terms of interpretation of the units of a derivative and interpreting those units into the real world meaning. However, I am going to spend our class time making sure that we have notation, terminology and a few definitions down.

Notation of the First Derivative

We have already seen the $f'(x)$ notation in the last section, but this is not the only notation for the first derivative. Another notation, introduced by a German mathematician named Leibniz is also used. It gets its basis from the definition of the first derivative as the change in y divided by the change in x. Change can be thought of as distances on the x and y axis and those distances are found by finding the differences in y and x values and Leibniz's notation reminds us of the **differences** in the y and x values (I'd like to add that the Greek letter delta in its lower case form also looks like a "d"). Here's what the notation looks like:

$f'(x) \approx \dfrac{\Delta y}{\Delta x}$ so, we use "d" to remind us of differences $f'(x) = \dfrac{dy}{dx}$

A formal way of interpreting this notation is: The derivative of y with respect to x.

$$f'(x) \text{ means } \frac{d}{dx}(y)$$

The Leibniz notation has some draw backs, mainly that it is difficult to indicate the derivative at certain values, and there the $f'(x)$ notation comes in quite handily.

Interpreting the First Derivative with Units

Recall that the units of the first derivative are those of the dependent divided by the independent (see the notation above to see this clearly). When interpreting this we see that the first derivative can be interpreted as the amount of change (of the dependent per unit (independent) at the given point (value of the independent). Pay attention to the book's Example #1on p. 102-103 because you will see that the first derivative is indicative of *marginal cost*.

Example: The time for a chemical reaction to happen, is given as time, T in minutes. The reaction time is a function of the amount of catalyst present, *a* in milliliters (mL). Thus $T = f(a)$. Use this to answer the following questions. *(#6p.106)

a) For $f(5) = 18$ what are the units of 5?

b) For $f(5) = 18$ what are the units of 18?

c) Interpret this statement in terms of the reaction.

d) What are the units of the derivative of $f(a)$?

e) Interpret $f'(5) = -3$ in terms of the reaction.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

The First Derivative to Estimate Function Values

Since the first derivative represents the amount of change at a given point. We can assume for points "relatively close" (small Δx's; $\Delta x \approx 0$ wrt the values of the function) to the given point that the function will continue to change by an amount that is approximately the same. This leads us to what is called a **Local Linear Approximation**, which is the approximate equation of the tangent line at the point.

$$\boxed{f(x) \approx f(a) + f'(a)\Delta x}$$

In other words, multiply the derivative by the change in x values and add it to the value of the function for which you have found the derivative.

Example: The quantity, Q in milligrams (mg) of nicotine in the body t minutes after smoking a cigarette is given by Q = f(t). Use this to answer the following questions. *(#30p.108)

a) Interpret f(20) = 0.36 in terms of the nicotine in the body. Use units.

b) Interpret f ′ (20) = -0.002 in terms of the nicotine in the body. Use units.

c) Use the information in part b) to estimate the amount of nicotine in the body at 21 minutes. Use units.

d) Use the information in part b) to estimate f(30). Use units.

The First Derivative and the Relative Rate of Change

We should be familiar with relative rates of change from algebra. These are our % increases and decreases. It is the ratio of the amount of change to the original/starting value. Here we see that means the derivative at some point *a* to the value of the function at that point.

$$\boxed{\text{Relative Change} = \frac{f'(a)}{f(a)}}$$

Example: The area of Brazil's rain forest, R = f(t), in million acres, is a function of the number of years since 2000, t. *(#42p. 109)

a) Interpret f(9) = 740 in terms of Brazil's rain forests. Use units.

b) Interpret f ′ (9) = -2.7 in terms of Brazil's rain forests. Use units.

c) What is the relative rate of change of f(t) when t = 9. Interpret.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§2.4 The Second Derivative

Your book has a very nice real world example describing the second derivative on p. 111, in the second paragraph. The **second derivative** describes the rate at which the slope is changing – is it increasing or decreasing in slope. The description your book gives involves the defense budget and the fact that the defense budget has been cut, but the budget is still increasing. It is merely, the rate of increase which has been slowed. It is the rate of increase that is the second derivative.

Here is a synopsis of the second derivative:

Notation: $f''(x)$

or

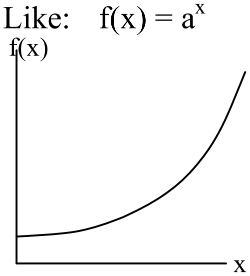
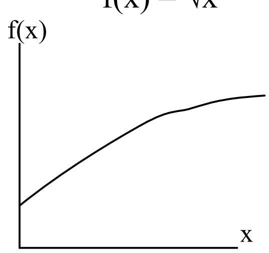
$\frac{d}{dx}\left(\frac{dy}{dx}\right)$

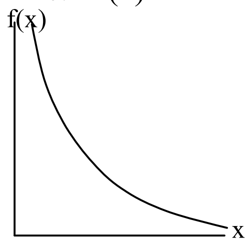
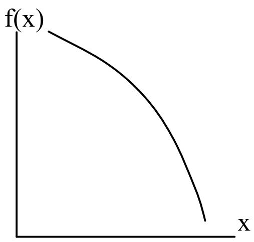
or

$\frac{d^2y}{dx^2}$

Meaning: The derivative of the $f'(x)$ (slope of the derivative function)
The concavity of f(x) (whether the slopes of tangent lines are getting > & > or < & <)

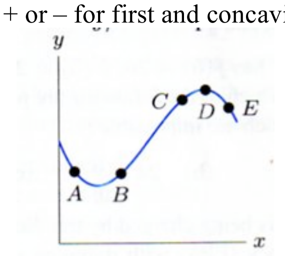
Here's a feel for the second derivative by examples with descriptions:

Original Function & Examples	**First Derivative**	**Second Derivative**
Like: $f(x) = a^x$ 	$f'(x) > 0$ Tangent lines have positive slope (increasing f(n)	$f''(x) > 0$ Tangent lines' slopes are getting bigger (> & >; concave up)
Like: $f(x) = -x^2$ for D: $(-\infty, 0)$ or $f(x) = \log x$ or $f(x) = -e^{-x}$ $f(x) = \sqrt{x}$ 	$f'(x) > 0$ Tangent lines have positive slope (increasing f(n)	$f''(x) < 0$ Tangent lines' slopes are getting smaller (< & <; concave down)

Original Function & Examples	First Derivative	Second Derivative
Like: $f(x) = a^{-x}$	$f'(x) < 0$	$f''(x) > 0$
	Tangent lines have negative slope (decreasing f(n))	Tangent lines' slopes are getting bigger (> & >; concave up)
Like: $f(x) = -x^2$ for D: $(0, \infty)$ or $f(x) = -e^{x}$ or $f(x) = \log(-x)$	$f'(x) < 0$	$f''(x) < 0$
	Tangent lines have negative slope (decreasing f(n)	Tangent lines' slopes are getting smaller (< & <; concave down)

Example: At which labeled points on the graph below are both $^{dy}/_{dx}$ and d^2y/dx^2 positive. *(#2p. 113)

Hint: Slopes + or – for first and concavity for second



Example: Sketch a graph of the function described in each part below. *(#9 p. 113)

a) First & second derivatives positive everywhere

b) Second derivative is negative everywhere & first derivative is positive everywhere

c) Second derivative is positive everywhere & first derivative is negative everywhere

d) First & second derivatives are negative everywhere

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Given the table below, answer the questions that follow:
*(#10 p.113)

t	100	110	120	130	140
w(t)	10.7	6.3	4.2	3.5	3.3

a) Does the derivative of the function appear to be positive or negative over the interval? Explain your answer.

b) Does the second derivative of the function appear to be positive or negative over the given interval? Explain your answer.

Example: IBM-Peru uses second derivatives to asses the relative success of various ad campaigns. They assume that all campaigns produce some increase in sales. Use this scenario to answer the following questions. *(#13 p.113)

a) What can be said about the sign of the first derivative of a graph of sales against time?

b) If a graph of sales against time shows a positive second derivative during a new ad campaign, what does this suggest to IBM management? Why?

c) What does a negative second derivative suggest to IBM management?

A note: Example 2 on p. 111 is worth some study. Here are some things to look at in terms of this example because we will be seeing them come up in the future. The point t* is a point that the concavity of the function changes, this is called an inflection point (a point where we go from increasing second derivatives to decreasing second derivatives or vice versa; interpretation that a rate of increase reaches a maximum and then begins to taper back or vice versa). The second thing of note in this example is idea of a limiting value. As the independent goes to a certain value, in this case infinity, the tangent lines slopes will reach a certain value. In this case it represents the maximum population that the environment can handle. In a business model it could represent the maximum price that could be charged for an item, etc.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

<u>§2.5 Marginal Cost and Revenue</u>

Now is the time to apply our new knowledge of derivatives to our "old" knowledge of cost and revenue from §1.4.

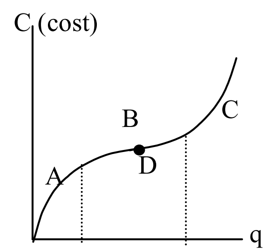
Recall:

Cost Function: C(q) = fixed $ + marginal $ x q
when Cost function is linear

Revenue Function: R(q) = p • q
when Revenue function is linear

Only now we acknowledge the fact that both cost and revenue functions can be non-linear. The cost function can have a <u>marginal cost</u> (recall this is the slope of the cost function) that is high at first, but then begins to decrease to a "close to steady" (just slightly increasing margin) rate until circumstances such as need to build a new factory or hirer more workers, etc. drive the marginal costs up at a faster rate. The marginal costs dropping off in their rate of increase is a concept known **<u>economy of scale</u>**.

Cost Function



Note: *In region "A" the slopes of the tangent lines are large and decreasing slowly. In region "B", the slopes are small and decreasing slightly to a point "D" where they begin to increase slightly. In region "C", the tangent lines' slopes are again getting large and increasing rapidly at first and then at slower rates. An investigation of the graph of the first derivative (recall our discussion in §2.2) can be quite useful in telling us about the <u>marginal cost</u> at different quantities!*

Example: If C(q) is the total cost of producing a quantity q of a certain product. Look at the figure above.* (#15p.120)

a) What is the meaning of C(0)?

b) Describe how the marginal cost changes as the quantity produced increases. (Hint: Use the first derivative. Graph C′(q) to help you and talk about the slopes of the tangent lines (the y-values) of the derivative function as q increases.)

c) Explain what the concavity tells you. (Hint: Think about the 2^{nd} derivative and what that is telling you about the change in the marginal cost. Concave down: marginal cost decreasing since slopes are decreasing.)

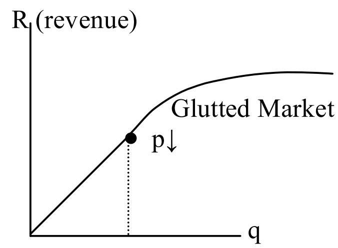
d) Explain what point D, the **<u>inflection point</u>**, the point where concavity changes, mean in terms of economics.

* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

e) In general, do you expect the graph of C(q) to look like this for all types of products? (Note: Don't take this part too literally. Not exactly like this, just the shape.)

The revenue function doesn't have to be linear either. It can be steady until a point and when the market is **glutted** – where the price falls due to consumer demand, the revenue function will look like the following.

Revenue Function



We did a problem back in §1.4, where we considered $\pi > 0$ or $R > C$ (like #4 p. 35). Although the book revisits this idea it is just to remind you that when $R > C$ there is a profit.

The "meat and potatoes" of this section incorporates our knowledge of derivatives and the literal interpretation of cost and revenue.

Many economic decisions are made based on the cost and revenue **"at the margin."** Decisions are made based upon whether greater additional cost or greater additional revenue is generated based on the production of the "nth + 1" item. (This should seem familiar, as the derivative is the change in slope from the nth to the nth + 1 point.)

Marginal Cost (MC) is additional cost at the "n^{th} + 1" item

➔ $C'(n) = C(n + 1) - C(n)$

Marginal Revenue (MR) is additional revenue at the "n^{th} + 1" item

➔ $R'(n) = R(n + 1) - R(n)$

Recall: In §2.3 the interpretation of the derivative ${}^{dy}/_{dx}$ at $x = n$ is the y to get the "n^{th} + 1" x. See Ex 2. p. 103 and note $f'(2000)$ is interpreted as the cost to get the 2001^{st} ton.

Example: The function C(q) gives the cost in dollars to produce q barrels of olive oil.*(#1p.119)

a) What are the units of the marginal cost?

b) What is the practical meaning of the statement MC = 3 for q = 100?

* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

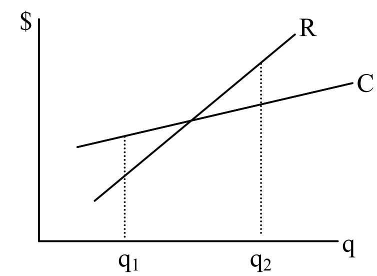
<u>Cost Analysis</u> is looking at the slopes of the tangent lines for costs and revenues and interpreting those slopes

Cost > Revenue when $C'(n) > R'(n)$

Revenue > Cost when $R'(n) > C'(n)$

When interpreting this we want to remember that the amount to produce the "$n^{th} + 1$" item is higher/lower than the income from the "$n^{th} + 1$" item

Example: Looking at the graph below showing a cost and revenue function for a car manufacturer, answer the following questions.* (#8 p. 119)



a) Which is greater at q_1, marginal cost or marginal revenue?

b) Which is greater at q_2, marginal cost or marginal revenue?

A comparison between profit and marginal revenues and costs, can indicate whether a company should increase production or decrease production. Here is a summary of the indicators:

If R > C then the company is currently making a profit

$\therefore$ If $R' > C'$ then production should be increased because at the next unit, the revenue will continue to be greater than the cost

and

If $R' < C'$, then production should be decreased because at the next unit, the cost will have become more than the revenue being generated and the company will lose money by making that additional item.

If R < C then the company is currently losing money (not making profit)

$\therefore$ If $R' > C'$ then production should be increased because at the next unit, the revenue will be greater than the cost, and the company will begin to make a profit

and

If $R' < C'$, then production should be decreased because at the next unit, the cost will have become even more and the company will lose more money than they are already l osing by making that additional item.

Example: At q_1 should the company increase or decrease production? What will the result be? What about at q_2?

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.