§1.1 What is a function?

A **function** is a set of ordered pairs for which every value of the **independent variable** (the values that can be inputted) has one and only one value for the **dependent variable** (the values that are output, dependent upon those input). All the possible values of the independent variable form the **domain** and the values given by the dependent variable form the **range**. *Think of a function as a machine and once a value is input it becomes something else, thus you can never input the same thing twice* and have it come out differently. This does not mean that you can't input different things and have them come out the same, however!

Is a the set of ordered pairs a *function*?

1) For every value in the domain is there only one value in the range?
   a) Looking at ordered pairs – If no x's repeat then it's a function (a map can be used to see this too. Domain on left & range on right, If any domain value has lines to more than one range value, then not a function.)
   b) Looking at a graph – **Vertical line test** (if any vertical line intersects the graph in more than one place the relation is not a function)
   c) Mathematical Model needs to consider the domain & range values or draw a picture– If input of any x will give different y's, then not a function (probably a graph is still best!)
   d) From a description – Try to model using a set of ordered pairs, a graph or a model to decide if it is a function.

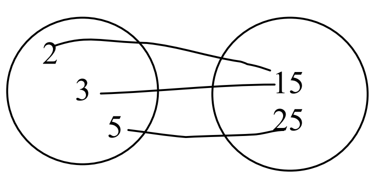
There are many ways to show a function. We can describe the function in words, draw a graph, list the domain and range values using set notation such as roster form or we can make a table of values, or we can use a mathematical model (an equation) to describe the function.

x	1	2	3
f(x)	2.3	2.8	3.2

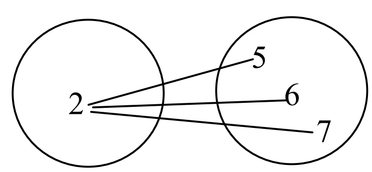
*(#11 p. 5)

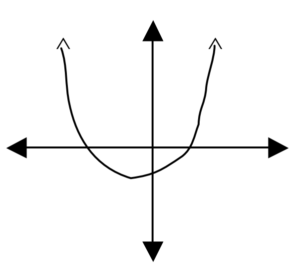
These are tables.
Left is f(n)
Rt. Is not an f(n)

x	1	1	2
f(x)	2.3	2.8	3.2

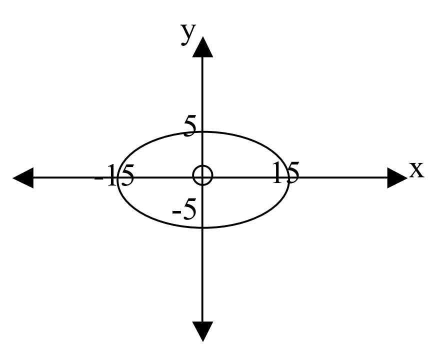


These are maps.
Left is f(n)
Rt. Is not an f(n)





These are graphs.
Left is f(n)
Rt. Is not an f(n)



* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

$y = 10x - x^2$

* (#8 p. 5)

These are models.
Left is f(n)
Rt. Is not an f(n)

$y^2 + x^2 = 9$

Although I won't give you an **example** that isn't a function, here an example in words that is a function. We'll investigate it by sketching a graph.

A patient experiencing rapid heart is administered a drug which causes the patient's heart rate to plunge dramatically and as the drug wears off, the patient's heart rate begins to slowly rise.

Function notation may have been discussed in algebra, but if it wasn't you didn't miss much. It is just a way of describing the dependent variable as a function of the independent. It is written using any letter, usually f or g and in parentheses the independent variable. This notation replaces the dependent variable, y.

f (x) ***Read as f of x***

The notation means evaluate the equation at the value given within the parentheses. It is exactly like saying y=!!

Example: The population of a city, P, in millions is a function *t*, the number of years since 1970, so P = f(*t*). Explain the meaning of the statement f(35) = 12 in terms of the population of the city.*(#2 p.5)

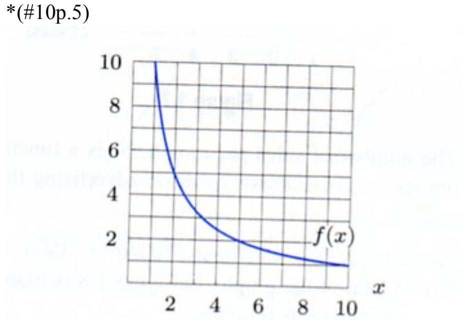
Example: Find the value of f(5)

a) $f(x) = 2x + 3$

b)

x	1	3	5	7
f(x)	9	12	15	18

c) *(#10p.5)



Function notation can also be used to solve equations.

Example: Find all values of x for which f(x) = 25 if $f(x) = x^2 + 9$

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

The following example shows how we can model a real life situation with a formula using 3 **parameters** (numbers that are subject to change), but once one parameter is firmly defined we create a linear function and then that function can be represented using function notation. Function notation gives some indication of the meaning of the independent and dependent variables. A graph of the function can be used visualize the relationship between the independent and dependent variables.

Example: The perimeter of a rectangle is P = 2L + 2w. If it is known that the length must be 10 feet, then the perimeter is a function of width.

a) Write this function using function notation

b) Find the perimeter given the width is 2 ft. Write this using function notation.

c) Use your graphing calculator to graph the function which shows the relationship between the length of the rectangle and the perimeter.

d) Use your calculator to find 3 sets of ordered pairs, writing those in a table here.

e) What do you notice happening to P(L) as L gets larger? What do you notice about the graph at L gets larger? This shows the concept of an **increasing function** – as the values of the independent increase so do the values of the independent.

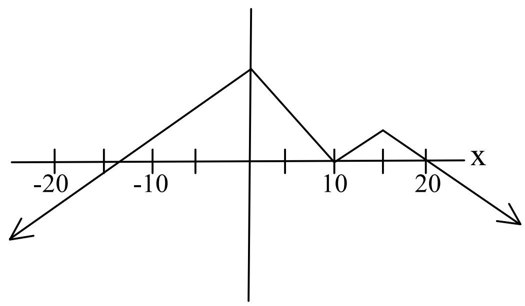
***Note:** We will see the decreasing function exhibited in an example for the next section.

Many times in this class we will only want to talk about what happens to a function over a certain interval of values. These values can be written using **interval notation**. It is essentially the shortcut of writing what you see on the number line – in other words you see real numbers on a number line, so it tells you about all real numbers between two endpoints. It tells about the inclusion or exclusion of left endpoint (**Endpoints** are the beginning or end of the solution set) and the right endpoint and slams them together with a comma in the middle. One way of describing all real numbers is the interval: $(-\infty, \infty)$. Infinity is elusive, since you can never reach it, and therefore in interval notation we always use a parenthesis around infinity.

Summary

Endpoint included	[or]
Endpoint not included	(or)
Negative Infinity	$(-\infty$
Positive Infinity	$\infty)$

Example: Using interval notation, give the intervals for which the independent variable on which the function is increasing.



<u>§1.2 Linear Functions w/ §1.4 Applications of Functions to Economics</u>

<u>Linear Equation in Two Variable</u> is an equation in the following form, whose *solutions are ordered pairs.* A straight line can graphically represent a linear equation in two variables. As long as we are not talking about a vertical line (x = any #), all linear equations are functions.

$$ax + by = c$$

a, b, & c are constants
x, y are variables
x & y both can't = 0

Also Recall from Beginning Algebra:
Solving an equation for y is called putting it in **<u>slope-intercept form</u>**. This is a special form, the functional form of the linear equation in two variables, which has the following properties.

$$y = mx + b \qquad \left.\begin{array}{l} m = \text{slope} \\ b = \text{y-intercept} \end{array}\right\} \text{Parameters}$$

The other great thing about this form is that it allows us to use function notation and eliminate the need to write the dependent variable. Hence, $y = mx + b$ becomes

$$f(x) = mx + b$$

since y is a function of x.

An <u>intercept</u> is where a graph crosses an axis (any graph, including those of linear functions have intercepts). There are two types of intercepts for any graph, <u>a horizontal intercept</u> (x-intercept;**<u>zeros</u>**) and a <u>vertical intercept</u> (y-intercept). A **horizontal intercept** is where the graph crosses the x-axis and it has an ordered pair of the form (x, 0). In terms of a function it describes at what value of the independent variable, the dependent variable reaches a value of zero. A **vertical intercept** is where the graph crosses the y-axis and it has an ordered pair of the form (0, y) most often written (0, b). In terms of a function it describes what the "base-line" value of the situation is. In other words, it gives the value of the dependent variable when the independent is zero.

<u>Finding the Y-intercept (X-intercept)</u>

Step 1: Let x = 0 (for x-intercept let y = 0)
Step 2: Solve the equation for y (solve for x to find the x-intercept)
Step 3: Form the ordered pair (0,y) where y is the solution from step two. [the ordered pair would be (x, 0)]

<u>Slope</u> is the ratio of vertical change to horizontal change. It is the rate of change of the dependent variable per unit of the independent. The last iteration of the slope presented here is referred to as the difference quotient (it is the same as the familiar $y_2 - y_1$ over $x_2 - x_1$ except it uses function notation).

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

<u>3Ways to Find Slope</u>

1) Formula given above

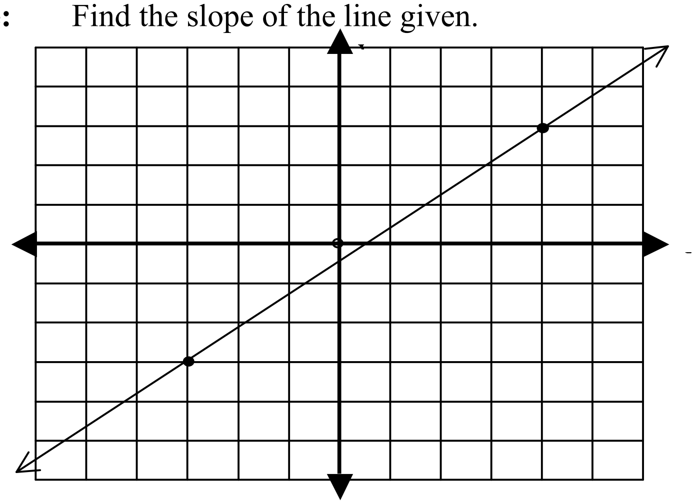
Example: Use the formula to find the slope of the line through

a) (4, 5) & (2, -1) b) (0, 0) & (1, 1)

2) Geometrically using m = $^{\text{rise}}/_{\text{run}}$

Choose points, create rise & run triangle, count & divide

Example: Find the slope of the line given.



* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

3) From the slope-intercept form of a linear function

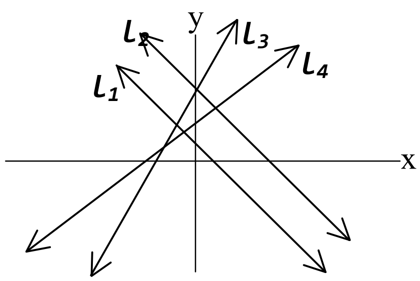
   $y = mx + b$, where m, the numeric coefficient of x is the slope

   Solve the equation for y, give numeric coeff. of x as the slope (including the sign)

   Example: Find the slope of the line $3x + 2y = 8$

Note: A family of functions represents a group of functions that share certain properties. By changing the parameters the function is "tweaked". Parallel lines form a family of functions. The parameter that is the y-intercept is changed, but everything else remains the same.

Example: Use the diagram below to answer the following questions



a) Which lines belong to the same family?

b) Of the lines in a), which has smaller slope?

c) Which of the lines in a) have the larger vertical intercept?

d) Which two lines have the same vertical intercept?

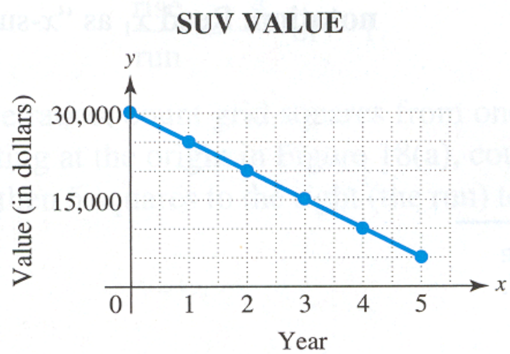
e) Which of the lines in d) has the larger slope?

f) Name two lines that represent decreasing functions.

The next example will show how a set of ordered pairs, which describes a real-life situation, such as the **depreciation function** (you'll find this in Ch. 1.4; the decline in value of an item per year from the starting price of the item) of a vehicle, can be used to give us information about a situation. This is the interpretation portion of the algebraic manipulation that we have been talking about.

Example: The following example came from p. 207, *Beginning Algebra,* 9th Edition, Lial, Hornsby and McGinnis

53. The graph shows the value of a certain sport utility vehicle over the first 5 yr of ownership.



Use the graph to do the following.

(a) Determine the initial value of the SUV. That's the y-intercept!

(b) Find the *depreciation* (loss in value) from the original value after the first 3 yr.

(c) What is the annual or yearly depreciation in each of the first 5 yr? Slope

(d) What does the ordered pair (5, 5000) mean in the context of this problem?

***Note:**. The baseline is the y-intercept and the "Amount per" is the slope. The slope is also known as a rate of change. It is the change in y divided by the change in x, so it is the rate at which the dependent is changing in increments of the independent variable. Many times we will be interested in the rate of change over time.*

(e) What would the horizontal intercept indicate in this situation?

(f) What happens to the values of the dependent variable as the independent variable increase? This shows the concept of a **<u>decreasing function</u>** – as the independent increases the independent values decrease.

Building a mathematical model of a situation that has a linear relationship requires knowledge of the slope and the vertical intercept. There are 2 ways to build the mathematical model given just two ordered pairs (two pieces of information relating the independent and dependent variables in two instances). We can use both the slope-intercept and the point-slope forms of a line to model data. To use the slope-intercept, we must know the base-line value (the value of the dependent when the independent is zero; the vertical intercept). We don't need the base-line if we use the point-slope form of a line to model.

<u>Point-Slope Form</u>

$\mathbf{y - y_0 = m(x - x_0)}$

m = slope

(x_0, y_0) is a point on the line

x & y are variables (don't substitute for those)

Now, let's use the two functional forms of a linear equation in two variables to find mathematical models for our two situations given above as slope problems.

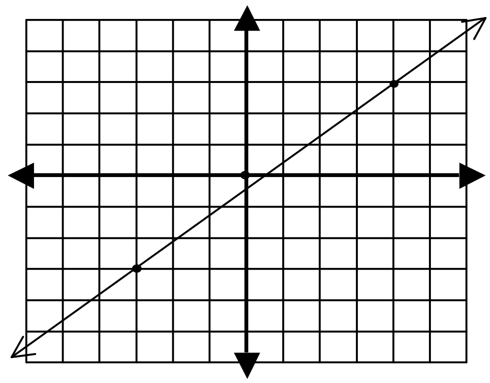
* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Find the equation of the lines described

a) Through the points *(*i)* is #8 p.12 & *ii)* is #6p.12)

i) (4, 5) & (2, -1) ii) (0, 0) & (1, 1)

b) Shown on the graph



But of course this is not how'll we be using it in this class. We want to do applications. For this we will borrow some applications from Economics found in section 4. We will be using the **cost function**, C(q), which is the total cost for quantity, q, of some good and the **revenue function**, R(q), which gives the total revenue received by a firm for selling a quantity, q, of some good and the **profit function**, π, given by R(q) – C(q).

Example: A company that makes jigsaw puzzles has fixed costs of $6000 plus each puzzle costs $2 per puzzle to make (the variable cost). The company sells each puzzle for $5 each.*(#14p.36)

a) Find the cost function (fixed costs are base-line amounts).

b) Find revenue function.

Example: Use the following table of to find the cost function.*(#12p.36)

q	0	5	10	15	20
C(q)	5000	5020	5040	5060	5080

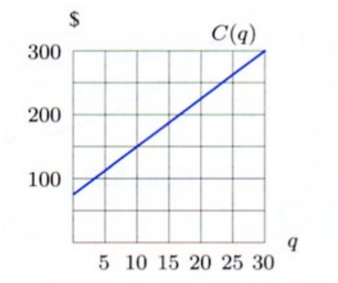
a) Using slope or vertical or horizontal intercept, which describes the fixed cost (cost to produce goods regardless of cost per item)? What is the fixed cost?

b) Using slope or vertical or horizontal intercept, which describes the cost to produce an item (the marginal cost)? What is the marginal cost?

c) Give the cost function.

* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

Example: Based on the figure below, answer the following questions.*(#3p.35)



a) Estimate the fixed cost of producing goods.

b) What is C(10)? What does this represent?

c) Using your estimate from a and your answer to b), what do you believe the marginal cost to be?

d) In relation to a line, what quantity does the marginal cost represent?

e) Give a cost function for this graph based upon the answers from the above questions?

This discussion leads naturally into section 1.4 material, so we will cover that section next, and since we have already started, there is no time like the present.

Break-even points are the point of intersection of a cost and a revenue function. The break-even point represents the quantity at which the revenue and the cost are equal and thus the **profit** is zero (recall that profit, π, is $R(q) - C(q)$). Recall from algebra that the intersection of two functions is where the independent and dependent value for both functions are identical. This point can be found mathematically or visually. Visually, we see the point of intersection of the functions' graphs and can read the values of the independent and dependent from the graph itself. Mathematically we can solve the system by setting the two functions equal and solving for the one variable (in the case of function notation we are solving for the independent value – in our case the quantity, q). Let's review the visual method using our calculators.

Method #1: Intersection of Equations (the solution of a system)

1) Graph each function
2) Find the intersection of the functions
3) The x-coordinate of the point of intersection is the quantity. This is the quantity at which the company will break even. The y-coordinate is the C(q) or the R(q), since it is the amount of money where $C(q)=R(q)$ or where $\pi(q)=0$.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Find the break-even point graphically. *(#4p.35)

$$C(q) = 6000 + 10q \qquad \& \qquad R(q) = 12q$$

Step 1: Find the key that looks like $\boxed{\text{Y=}}$ and push it.
Step 2: Using the $\boxed{\text{X,T,}\theta\text{,n}}$ key and the $\boxed{(-)}$ and $\boxed{+}$ key in the equations to Y_1 and Y_2 (you can move between those with the arrow keys)
Step 3: Find the $\boxed{\text{ZOOM}}$ key and choose Standard (use arrows or enter 6) This graphs the 2 equations.
Step 4: $\boxed{2^{nd}}$ $\boxed{\text{TRACE}}$ will get you into the CALC menu, and you need the INTERSECT function (use arrows or enter 5). Once there, Y_1 should be in the upper left corner, if it isn't then down arrow until it is and $\boxed{\text{ENTER}}$ Now, Y_2 should be in the upper left corner, press $\boxed{\text{ENTER}}$ again. It will now say GUESS in the lower left corner, press $\boxed{\text{ENTER}}$ again and your intersection will be shown.

Interpretation: The intersection point represents the **break-even point**, so that means the quantity, q, at which the money to produce [C(q)] will be equal to the money made [R(q)], or put another way, where the **profit** is zero[$\pi(q)=0$].

b) How many units must be produced for the company to break even? Write the break-even point as an ordered pair using correct units for the dependent variable.

Recall from algebra that there are two ways of finding the solution to a linear equation in one variable graphically. One method is by treating a linear equation in one variable like the equality of two equations so that the end result is as we saw in the last example. The other method is to solve the equation so that you have an equation that reads $0 = f(x)$ and then to graph it and find the x-intercept. When taking algebra we seldom have any clue why we'd be interested in this, but now we have just an instance where we are interested! If we take the equation we create by setting $C(q) = R(q)$ and push all quantities to one side achieving $0 = f(x)$, we will have produced the **profit function**! The *0=f(x)* therefore means at what quantity (the x in this case being q) is the *profit = zero*. Now, let's try our solution above in this manner and also get our first glimpse of a **profit function**. You should notice that the x-intercept is the solution, the break-even point, and it agrees with the x-coordinate of the above (just as you were taught in algebra).

<u>Mehod #2: X-Intercept Method</u>

1) Move all terms to either the left or right using addition property of equality
2) Graph the equation and locate the x-intercept. The x-coordinate is the solution.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Give the profit function and quantity produced at which the company will break even using the x-intercept method. Use the same C(q) & R(q) from above.

Step 1: Set C(q)=R(q) and push all terms to one side so you attain 0 = f(q

Step 2: Again graph the equation. **Note:** If there are other equations in your calculator you can prevent them from being graphed by moving your cursor to the equal sign (use the arrow keys) and pressing $\boxed{\text{ENTER}}$ The equal signs will be un-highlighted, meaning that they won't be graphed.

Step 3: Again go to the CALCULATE menu (see the above) and this time choose the ZERO (use the arrow keys or press 2). The calculator will prompt LOWER BOUND? in the lower left corner and you should press $\boxed{\text{ENTER}}$ Now it (use the up/down arrow key) to move the cursor along the line until it is on the other side of the x-intercept, and press $\boxed{\text{ENTER}}$ again. Now it will prompt you with GUESS? in the lower left corner and you will press $\boxed{\text{ENTER}}$ again and be rewarded with the x-intercept.

Your Turn

Example: Let's return to the puzzle problem from above.*(#14p.36) Using the equations that we found, find:

a) The profit function.

b) The quantity produced to break-even

c) Give the ordered pair that represents the break-even point

d) What is the **marginal revenue** (just like marginal cost, it is a rate of change for the revenue, and therefore is the slope of the revenue line)?

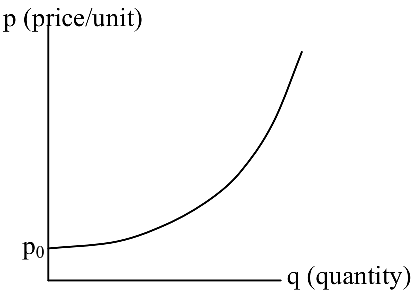
e) What is the **marginal profit** (just like marginal cost, it is a rate of change for the profit, and therefore is the slope of the profit line)?

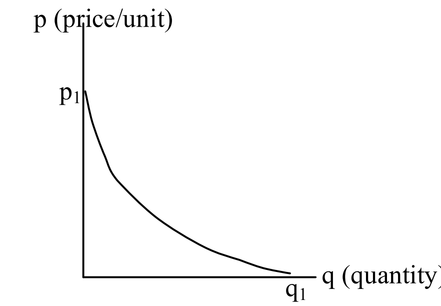
According to Wikipedia: **Supply and demand** is an economic model of price determination in a market. It concludes that in a competitive market, the unit price for a particular good will vary until it settles at a point where the quantity demanded by consumers (at current price) will equal the quantity supplied by producers (at current price), resulting in an economic equilibrium of price and quantity.

The long and short of it as supply increases, demand decreases and vice versa and at some point the two curves intersect at a point called the **equilibrium price** (like dependent value of the break-even point). First let's consider the equations for supply and demand and the meaning of the slopes and the intercepts. After that we can consider what happens when taxes are imposed, whether they are a **specific tax** (fixed amount imposed on producer per unit) or **sales tax** (fixed percentage imposed on sale price) the equilibrium price is shifted.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Supply Curve | **Demand Curve**



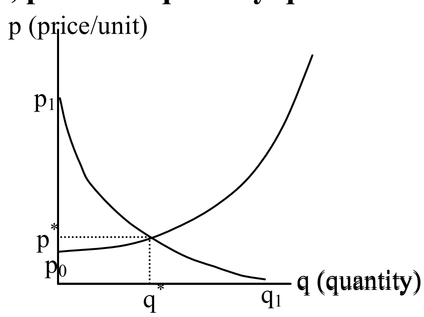


S = Amt Rec'd • q – p_0

D = p_1 – Amt Paid • q

Where p_0 is the price at which the company won't/can't produce any units
p_1 is the price at which the consumer won't buy any units
q_1 is the quantity which the consumers will consume if the item is free

Visual of the Equilibrium Price, p* at the quantity q*



Although supply and demand curves are actually curves and not linear functions, this section uses a linear approximation for the supply and demand "curves".

Example: The supply curve is given by S(q) = 2500 – 20p while the demand curve is given by D(q) = 10p – 500.*(p.38#37)

a) What is the **equilibrium price**?

b) Graph the supply and demand curves and plot the equilibrium price (with its quantity).

c) A specific tax (amt. rec'd is reduced in the supply curve for a specific tax) of \$6 is applied per unit sold. Find the new equation of the supply curve and graph it as S'(q).

d) What is the new equilibrium price?

e) What portion of the specific tax is passed on to the consumer (this is the difference between the p*' and p*).

f) If a sales tax (amt paid by consumer is increased by the percent of the amt paid, thus 1.0x for x%) of 5% is imposed on the consumer, write a new equation for the demand curve, D'(s).

g) Based on the new demand curve and the original supply curve, what is the equilibrium price?

h) Just as with the price when a specific tax is imposed, the consumer

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

and producer share the cost and the government benefits. The original price minus the new equilibrium price is the amount of tax paid by the supplier. Find the amount of tax paid per unit by the supplier.

i) The consumer gets the short-end of the stick and must pay 1.0x (x being x%) of the new equilibrium price in sales tax. Calculate the tax per unit paid by the consumer.

j) The government reaps the benefit and gets the money paid by both the consumer and producer (answers in h&i) per unit. What is the amount that the government gets per unit?

k) Finally, the total tax collected by the government will be the total tax paid per unit by the consumer and producer (answer in j) government?

§1.3 Average Rate of Change and Relative Change

The **rate of change** is the slope of a function. You should be able to find the rate of change for any function given in any form (visual/tabular/formula/description). Recall that slopes tell us if the dependent is increasing or decreasing as the independent is increasing, thus they are telling us if a function is increasing or decreasing.

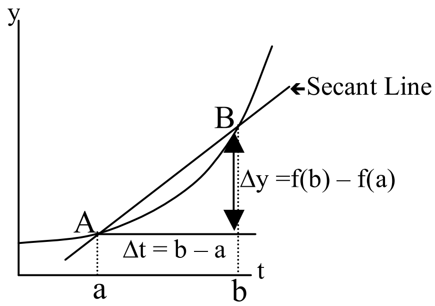
positive slope ➔ increasing dependents

negative slope ➔ decreasing dependents

Unfortunately, not all functions are linear and the rate of change is not always constant. for **non-linear functions** we can find an **average rate of change** which is the rate of change on an interval of time. This brings about the idea of a **secant line** – a line created by two points on a curve that tells us about the ***average rate of change*** on an interval.

Ave. Rate of Change $= \dfrac{f(b) - f(a)}{b - a} = \dfrac{\Delta y}{\Delta t}$ for $t = b$ and $t = a$ when $y = f(t)$

Visual of Average Rate of Change



Secant Line

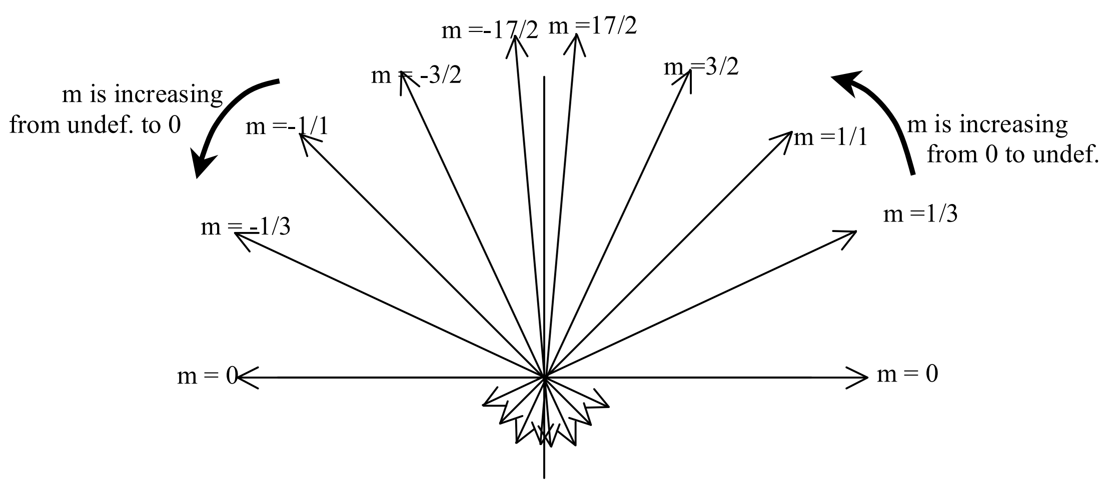
Slope ➔ if + then **increasing** function
− then **decreasing** function

Location (in Relation to Curve) ➔ if above then **concave up**
if below then **concave down**

Slopes of successive secant lines
If **increasing** (> and > slopes), then **concave up****
If **decreasing** (< and < slopes), then **concave down****

****Note:** A straight line is neither concave up or concave down as it has constant rate of change(slope)

Let's look at a visualization of increasing slopes:

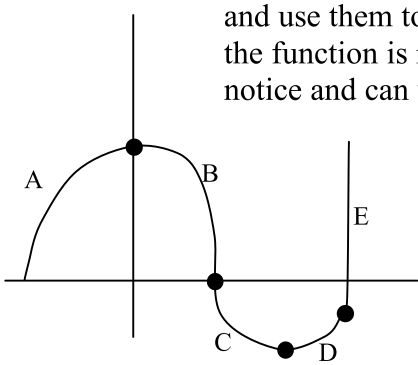


Why do we care if a function is increasing or decreasing and whether it is concave up or down?

Increasing/Decreasing ➔ Interpretation of what happens to the dependent over time (the independent being time)

Concavity ➔ Rate of change (slope) increasing or decreasing over time

Example: For the following graph, draw three secant lines in regions A & B and use them to discuss the concavity of the function and whether the function is increasing or decreasing in the region. What do you notice and can you say about region E?



Before we begin examples, we should note an important definition that is used in application problems in this section. **Velocity** is an average rate of change of distance wrt (with respect to) time. Velocity is different from *speed* in that velocity has direction as well as magnitude. In other words we have positive and negative velocities. **Speed** is just the magnitude – the size without taking into account direction. Positive velocities can be interpreted as speeding up if we consider an object at rest that is then given acceleration and negative as brakes are applied. It can also be interpreted as positive as the object slows down and negative as it increases as in the case of an object put into motion and then acted upon by gravity.

Let's consider the example 8 given by your book on p. 20 and take it one step further.

Example: The height of the grapefruit t seconds after it is thrown in the air is shown by the table:*(p. 20 Table 1.9 Ex. 8)

t (sec)	0	1	2	3	4	5	6
f(t) (ft)	6	90	142	162	150	106	30

Example 8 asks you to calculate the average velocity of the grapefruit between t=4 and t=6. Let's take this one step further to show concavity and what it tells us about velocity.

a) Calculate the slope between t=2 and t=3 and interpret the sign of the in terms of the change in height.

b) Calculate the slope between t = 3 & t = 4
c) Calculate the slope between t=5 & t = 6
d) Based on the slopes calculated in b & c what is the concavity of the curve? How do you know?
e) What does the concavity indicate about velocity?
f) Use the sign of the slope and the concavity to sketch a graph.

We've seen from the example above how increasing vs. decreasing slope of a secant line can tell us about the increase/decrease of the dependent value and how the secant line can show us concavity using the slope.

* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

Example: Using the secant line's slope, sketch a graph of the following situation.

A car starts slowly and then speeds up. Eventually the car slows down and stops. *(#40p.26)

§1.5 Exponential Functions

An **exponential function** is any positive $\Re$ not equal to one raised to some $\Re$ valued exponent.

$$P = f(t) = P_0\, a^t \quad \text{where } a > 0\text{, but } a \neq 0$$

The **domain** of the function, the values for which x can equal, are all $\Re$ $(-\infty,\infty)$.

Why? Because an exponent can be any real number. And the exponent is the independent variable.

Note: *For our applications we will only be interested in D: $[0, \infty)$*

The **range** however, is all positive $\Re$ greater than zero $(0, \infty)$.

Why? Because any number raised to some power will always be a positive number.

The quantity P_0 represents the **initial quantity** (the vertical intercept)

Why? Because when $t = 0$ that makes "a^t" equal to one, which gives P_0

For **exponential growth** $a > 1$, giving the increasing function you see below on the left

$a = 1 + r$ where r = percent rate of change; growth is **positive rate of change**

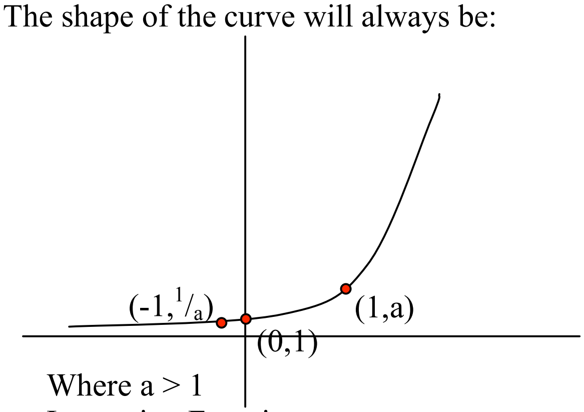
For **exponential decay** $0 < a < 1$, giving the decreasing function you see below on the rt.

$a = 1 + r$ where r = percent rate of change; it is **negative for decay**

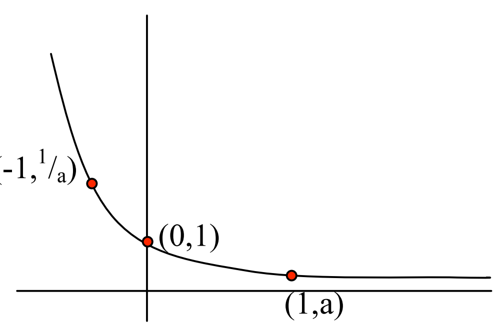
To **recognize** an exponential function from values, look for **ratios** of P values that are **constantly spaced** for constant intervals of t

To **create** an exponential equation, find the **constant ratio**, raise it **to the power of t** and **multiply** it **by** the **initial** value.

The shape of the curve will always be:



Where $a > 1$
Increasing Function



Where $0 < a < 1$
Decreasing Function

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Just some notes:

1) An exponential graph will always approach the x-axis but will never touch the axis.
2) The graph will grow larger and larger if the number (a) is greater than 1. This is like population. Think of 2 people, they have children, then those have children and so on, this is an exponential function, and you should be able to see that it grows to an infinite number.
3) If the a is between one and zero (fractional; decimal less than one but greater than zero) the shape is the same but starts at infinity and decreases to approach the x-axis.

Now, let's see if we have the basics of the exponential function.

Example: The following functions give the amount of substance present at time t. In each, give i) the amount present initially, ii) state whether the function represents exponential growth or decay, iii) and give the percent growth or decay rate.*(#2 p.43)

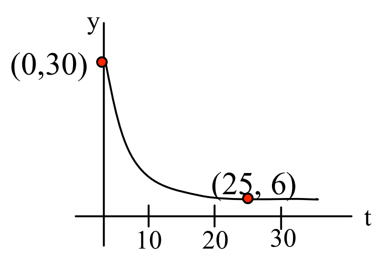
a) $A = 100(1.07)^t$ b) $A = 3500(0.98)^t$

Example: A town has a population of 1000 people at t = 0. In each of the following cases, write a formula for the population, P, of the town as a function of year t.*(#4 p. 44)

a) The population increases by 50 people a year
b) The population increases by 5% per year

Note: *One of these is not linear.*

Example: Find a possible formula for the function. *(#18 p. 45)



Note: *Write the equation as you have it and solve for the percent rate of change.*

Example: For the data in the table showing annual soybean production in millions of tons, answer the following questions.*(#26 p.45)

Year	2000	2001	2002	2003	2004	2005
Production	161.0	170.3	180.2	190.7	201.8	213.5

a) Does this table correspond to a linear or exponential function?
b) Find a formula for the the world soybean production in millions of tons, as a function of time, t, since 2000.
c) What is the annual percent increase in soybean production?

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§1.6 The Natural Logarithm

When we want to solve for a missing factor, we do division. When we want to solve for a missing addend we do subtraction. These are the inverse operations. When we want to solve for a missing exponent we will also need its inverse. The inverse of an exponential is a logarithm. We will be discussing the inverse of an exponential with the base *e* in this section.

Quick review of inverse functions facts:

1) A function must be one to one for an inverse to exist
2) Inverse functions domains and ranges switch
   a) D ➔ R and R ➔ D
   b) Ordered pairs (x, y) ➔ (y, x)
3) Graphically an function and its inverse are symmetric across the line y = x

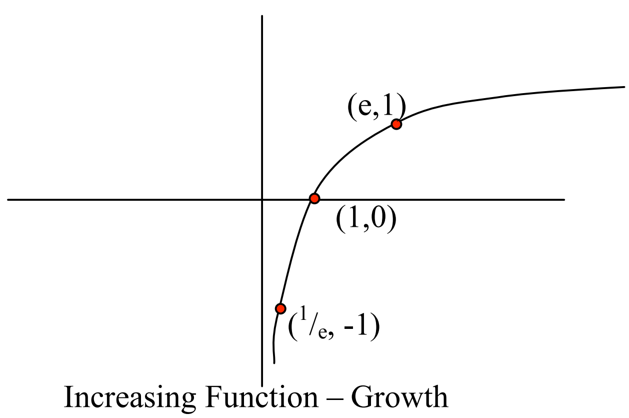
The Natural Logarithm

$y = \log_e x$ $e \approx 2.71828183$ which is written as $y = \ln x$

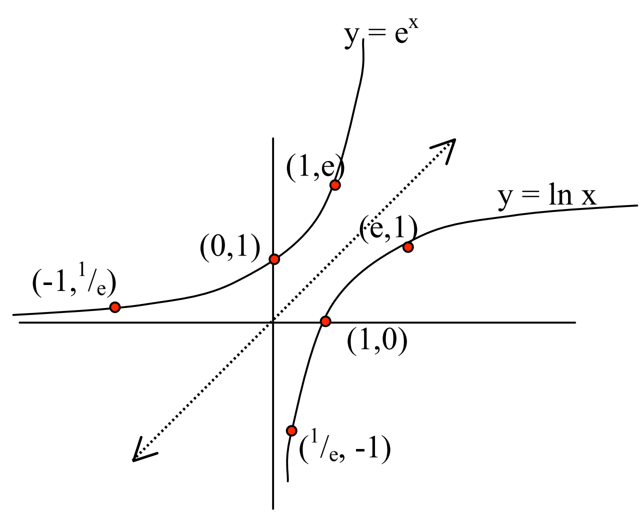
D: $(0, \infty)$ & R: $(-\infty, \infty)$

converts to exponential $x = e^y$
(you'll see $a = e^k$ in your applied problems)

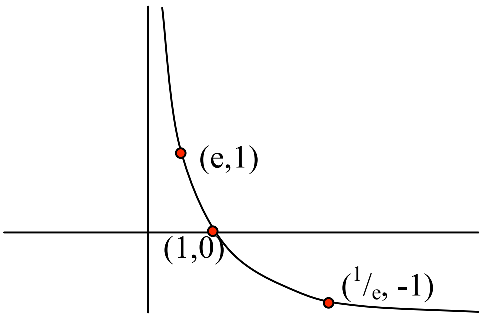
b/c the exponential and log are inverses the domain and the range inverts; can be achieved by taking e to the power of each side and applying rule 5 below

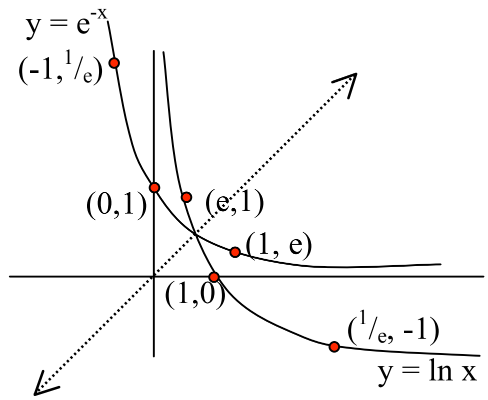


Increasing Function – Growth



* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.





Decreasing Function -- Decay

<u>Properties of Logarithms</u>

1)	$\ln AB = \ln A + \ln B$	➔	Known as the product rule of logarithms
2)	$\ln A/B = \ln A - \ln B$	➔	Known as the quotient rule of logarithms
3)	$\ln A^p = p \ln A$	➔	Known as the power rule of logarithms
4)	$\ln e^x = x$	➔	If the base & **<u>argument</u>** (the quantity for which the log is found)are the same the answer is the exponent of the *argument*
5)	$e^{\ln x} = x$	➔	If the base and the base of the log in the exponent are the same then the answer is the argument

We will use these properties to solve exponential equations, because in order to get a solution for an exponent we have to be able to "get at the exponent" which may first require property number 4 and could also potentially be require the use of properties 1-3 to isolate the variable.

Example: Find *t*.*(b is #6, c is #16 p. 50)

a) $25 = 5e^t$ b) $100 = 25(1.5)^t$ c) $7 \cdot 3^t = 5 \cdot 2^t$

Just as in the last section, we will be interested in ***initial population*** and ***annual growth rate/decay rate.*** But in addition to ***annual growth/decay rate*** we will also be interested in **continuous yearly growth/decay rate** – given by the numeric coefficient of *t* when we have the base *e* instead of *a*. Thus we will want the ln equation in the following form:

$P = P_0a^t = P_0(e^k)^t = P_0e^{kt}$ so, $a = e^k \therefore k = \ln a = \ln(e^k)$ where
k is the continuous growth rate
$a = e^k$ where $a = 1 + r$ **and**
r is growth rate per unit time

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Let's practice the skill of manipulation first and then we can apply it more readily.

Example: Give the growth or decay rate and state if it is continuous.
*(a is #18 & b is #20p. 50)

a) $P = 7.7(0.92)^t$ b) $P = 3.2e^{0.03t}$

Example: Write in $P = P_0a^t$ form and indicate if it represents growth or decay.*(#24 p.50)

$$P = 2e^{-0.5t}$$

Example: Write in $P = P_0e^{kt}$ form and indicate if it represents growth or decay.*(#28 p.50)

$$P = 10(1.7)^t$$

Example: For the function $P = 100e^{0.06t}$, which represents a population size after t years.
*(a-c are from #34p.50)

a) What is the continuous growth rate?
b) Write the function in terms of $P = P_0a^t$
c) What is the annual growth rate per year?
d) What is the initial population?
e) To the nearest unit, give the population, P, after 10 years.
f) To the nearest 10^{th} of a year, when will the population, P, reach 500?
g) The amount of time it takes for a population to double in size is called the **doubling time**. This population will double when it reaches what size?
h) What is the doubling time for this population to the nearest tenth of a year?

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: A city's population is 1000 and it is growing by 5% per year.*(a- c are from #22p.50)

a) Find a formula for the population at time t years from now is an annual rate of growth.

b) Find a formula for the population at time t years from now is an continuous rate of growth.

c) Estimate the population of the city in 10 years given a).

d) Estimate the amount of time, to the nearest year, for the population to double given b).

<u>§1.7 Exponential Growth and Decay</u>

This section is just the application of the previous sections using examples of both growth and decay. You will be asked to solve for the dependent as well as finding continuous growth/decay rates (solving for a value in the exponent).

Another discussion of doubling time takes place and comparable **<u>half-life</u>** (comparable to doubling time for a function exhibiting decay). It is shown in this section, that it doesn't matter what the initial quantity or growth/decay rates in the doubling time and the half-life, these are a constant equivalent to ln (2) and ln ($^1/_2$), respectively. See Ex. 3 & 4 on page 52 for further enlightenment.

A discussion of application of annual and continuous growth rates are exhibited through a financial example – Annually compounded interest and Continually compound interest.

Compounded Annually	$P = P_0(1 + r)^t$
Compounded Continuously	$P = P_0e^{rt}$

Finally, there is a discussion from economics, on **<u>present</u>** and **<u>future</u> <u>values</u>**. This discussion concerns the concept that it is better to have money now than in the future, and therefore if you are willing to take the money in the future, how much compensation should be given for the loss of the potential earning should be expected. It is important to note, that *from this perspective*, you ***consider <u>present value</u> to compare best-case scenarios***! This is the one I will focus the most attention on in this section.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

B = Future Value of a payment, P, to which P would have grown t time from present.

P = Present Value of a future payment, B, if it had been paid to you at the present time instead of t time from the present.

***Note:** These each consider having invested the money in an interest bearing account yielding either annual percentage rate or a continuously compounded rate.*

$B = P(1+r)^t$	or	$B = Pe^{rt}$
	and	
$P = B(1+r)^{-t}$	or	$P = Be^{-rt}$

Example: What is the future value in 8 years of a \$10,000 payment today, if the interest rate is 3% per year compounded continuously?*(#32p.58)

Example: What is the present value of an \$8,000 payment to be made in 5 years, if the interest rate is 4% per year compounded continuously?*(#34p.58)

Example: If interest is compounded annually. Consider the following choices of payments to you:*(#36p.58)

Choice 1: \$1500 now and \$3000 one year from now
Choice 2: \$1900 now and \$2500 one year from now

a) If the interest rate on the savings were 5% per year, which would you prefer (remember you are considering present value to answer this question)?
b) Is there an interest rate that would lead you to make a different choice?

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

§1.8 New Functions from Old

Recall the Algebra of Functions

$$f(x) + g(x) = (f + g)(x)$$
$$f(x) - g(x) = (f - g)(x)$$
$$f(x) \bullet g(x) = (f \bullet g)(x)$$
$$^{f(x)}/_{g(x)} = (^{f}/_{g})(x) \quad \text{if } g(x) \neq 0$$

Summary: Functions can added, subtracted, multiplied, or divided. This can be accomplished by adding/subtracting/multiplying/dividing the function in general or by adding/subtracting/multiplying/dividing the values of the functions.

A **composite function** is a function evaluated with/at another function.
Note: *This may not look like another function, but x + 1 can be considered a function of x.*

$f[g(x)]$ is a composite function, f with g

Summary: This is a substitution problem. Let the value of the outside function's independent be the expression representing the inside function.

Example: For $f(x) = x^2$ and $g(x) = 3x - 1$ *(#7 p. 62) find each of the following.

a) $f(2) + g(2)$ b) $f(2) \bullet g(2)$ c) $f(g(2))$

d) $g(f(2))$ e) $f(x + h) - g(x)$ f) $f(g(x))$

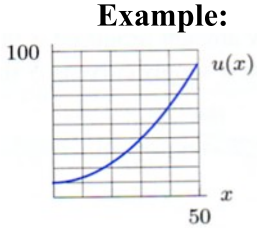
Note: *Once you find a function in general, like f), it is easy to find it for specific values. Try finding c) based upon your answer for f). You should get the same answer!*

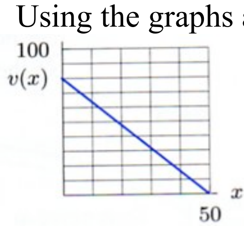
Example: Finish the table for the following function. *(#11 &12p. 62)

x	0	1	2	3	4	5
f(x)	10	6	3	4	7	11
g(x)	2	3	5	8	12	15

a) Using the table find $g(f(2))$
b) Using the table find $f(g(2))$
c) Using the table find $f(1) \bullet g(1)$
d) Using the table find $g(2) - f(2)$

Example: Using the graphs approximate the values requested.*(#25&26 p. 62)





a) Find $u(v(40))$

b) Find $v(u(10))$

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

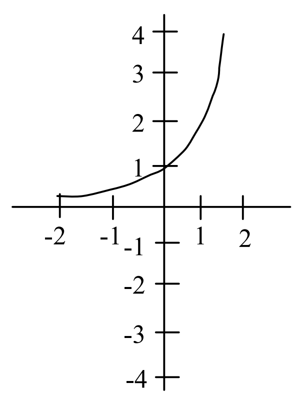
<u>Translations of Functions</u> move a function around in space. Remember the idea of a family of functions? This is what translations do – they create families. We can translate a function in several ways: stretching (multiplication of the function by a positive constant), reflecting (multiplying the function by a negative), horizontally (subtracting a constant from the independent), vertically (adding a constant to the function). Here is a summary of these translations in function notation.

Stretching:	If y = f(x), then the reflected function is where "c" is some positive constant	y = cf(x)
Reflection:	If y = f(x), then the reflected function is	y = -f(x)
Horizontal:	If y = f(x), then the reflected function is where "k" is some constant by which the function is shifted. If k is positive it is right and if k is negative it is left. Note that it is read from "x – k"	y = f(x – k),
Vertical:	If y = f(x), then the reflected function is where "k" is some constant by which the function is shifted up or down. If k is positive it is shifted up and if k is negative it is shifted down.	y = f(x) + k

Note: *I have listed the translations in the order that we generally think of moving a function about the coordinate system. We start with the based shape at the zero position and then we consider the movement of that function translation by translation.*

Example: We will take the following function through the following translations to exhibit, translations of functions. *(#30p.63)

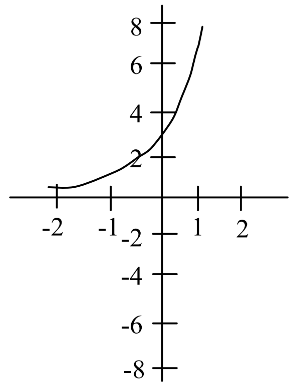
x	-1	0	1
f(x)	$^1/_2$	1	2



* *Applied Calculus*, Hughes-Hallett et al, 4^{th} Edition, Wiley, 2010.

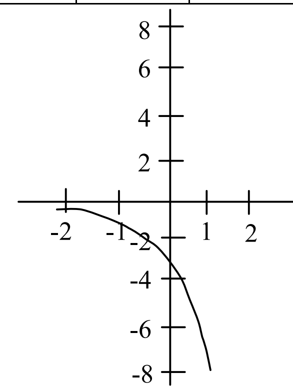
a) $y' = 3f(x)$

x	-1	0	1
f(x)	$^{3}/_{2}$	3	6



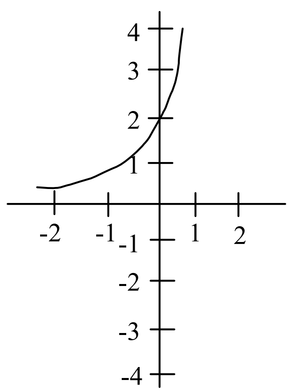
b) $y' = -3f(x)$

x	-1	0	1
f(x)	$^{-3}/_{2}$	-3	-6



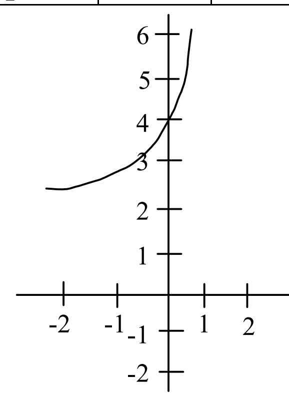
c) $y' = f(x + 1)$

x	-2	-1	0
f(x)	$^{1}/_{2}$	1	2



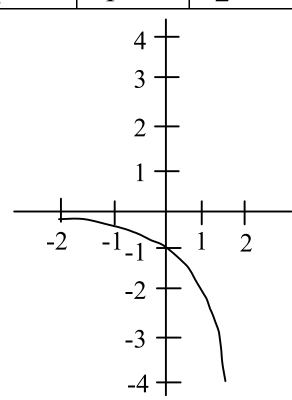
d) $y' = f(x + 1) + 2$

x	-2	-1	0
f(x)	$^{5}/_{2}$	3	4



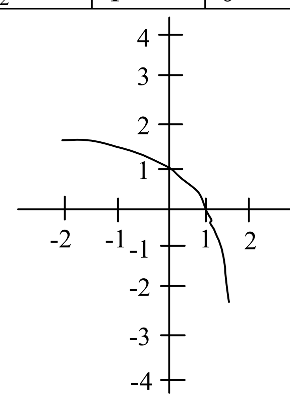
e) $y' = -f(x)$

x	-1	0	1
f(x)	$^{-1}/_{2}$	-1	-2



f) $y' = -f(x) + 2$

x	-1	0	1
f(x)	$^{3}/_{2}$	1	0



* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Note: I've changed the example slightly from the exercise in the book and added tables of values so that we can see how the functions are changed both visually and numerically by translations. When we do translations, there is a specific order that we think of them in: 1) Stretching, 2) Reflection, 3) Horizontal, 4) Vertical. The Stretching multiplies the f(x) value by the constant. The Reflection takes the opposite of the stretched f(x) value. The Horizontal does nothing to f(x), it simply changes the x value by subtracting the constant (must be in form x – c to see what to subtract). The Vertical adds the constant to the stretched, reflected f(x) value.

§1.9 Power Functions

A **power function** is any function where the independent is raised to a power that can be written in the form. We can say that y is **proportional** to x^p:

$y = kx^p$ where k = constant, called the constant of proportionality
p is the power **and**
x is the independent variable

Note: *If $y = kx^{-p}$ then y **is inversely proportional** to x^p. In other words, the constant of proportionality, $k = y \cdot x^p$*

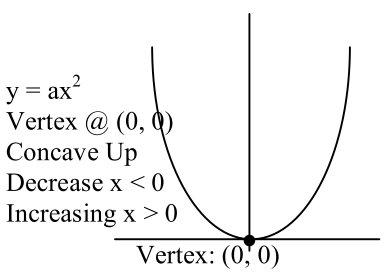
Some families of power functions to be familiar with:

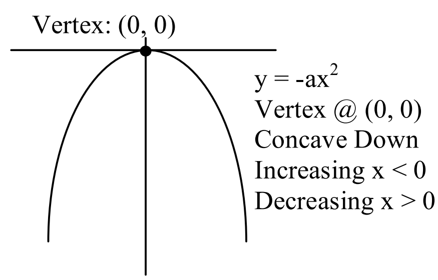
Quadratic: $y = ax^2 + bx + c$

Sign of a determines opens up or down
"+" concave up & increasing on negative x while increasing on positive x
"–" concave down & increasing on negative x while decreasing on neg. x

The **vertex** (where the graph changes direction) is at $(-{}^{b}/_{2a}, f(-{}^{b}/_{2a}))$

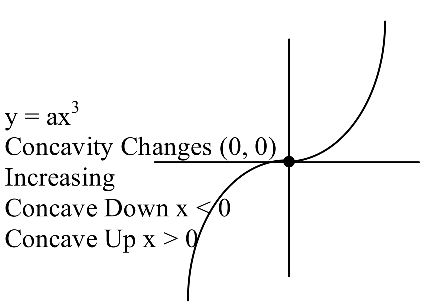
Symmetric around a vertical line called a **line of symmetry**
Goes through the vertex: $x = -{}^{b}/_{2a}$

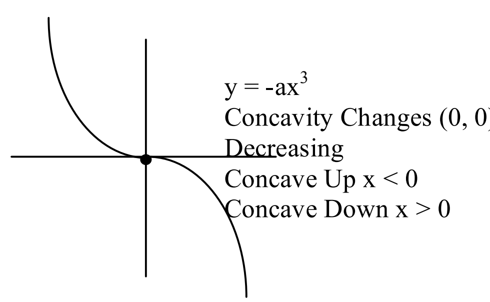




Cubic: $y = ax^3 + bx^2 + cx + d$

Form a lazy "S" shaped graph
Sign of "a" determines curves up and to right or down and to right
"+" increasing & concave down on neg. x and concave down on pos. x
"–" decreasing & concave up on neg x and concave down on pos. x

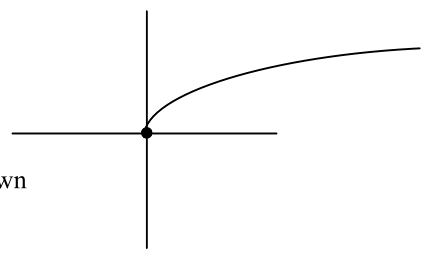


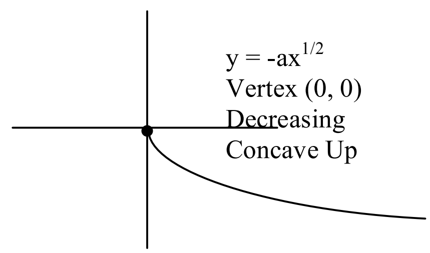


> A special note about the last two functions (one we'll talk about later):
> $y = x^2$ is always increasing as x is increasing so we say
> $x^2 \to \infty$ as $x \to \infty$
> $y = x^3$ is always decreasing as x is decreasing so we say
> $x^3 \to -\infty$ as $x \to -\infty$

Square Root: $y = a(x - b)^{1/2} + c$

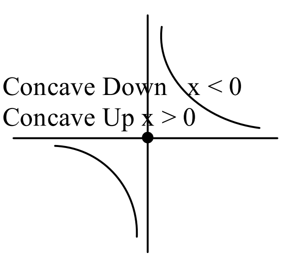
Looks like half a parabola opening up or down from the x-axis
Only half because domain is range of its inverse the quadratic, so the domain is $(0, \infty)$ for the basic family graphed below
Sign of "a" tells up or down from x-axis
"+" is increasing & concave down
"–" is decreasing & concave up
Vertex is (b, c)

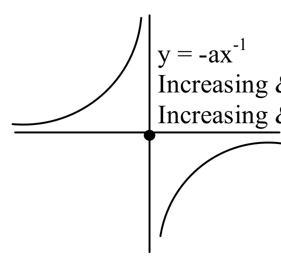




Inverse: $y = a(x - b)^{-1} + c$

Look like a wide parabola in 1st & 3rd or in 2nd & 4th quadrants
Domain must exclude "zero"
∴Never touches or crosses an axis; either the x or y (forming asymptotes)
Because $x \neq 0$, but as it gets close it is very small which makes f(x) get big
Like wise as x gets big, f(x) gets small but never gets to zero!
Sign of "a" tells which quadrants
"+" decreasing & concave down on $x<0$ & concave up on $x>0$
"–" increasing & concave up on $x<0$ & concave down on $x>0$





Now, that we've gone through the families of power functions and discussed proportionality, we should do some problems.

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.

Example: Answer the following questions for each of the parts.*(#2-5p.67)

i) Is the function a power function?

ii) If so, then write in the form $y = kx^p$

iii) Give the value of k & p

a) $y = \dfrac{3}{x}$ b) $y = 2^x$ c) $y = \dfrac{3}{8x}$

d) $y = (3x^5)^2$

Example: The strength, S, of a beam is proportional to the square of its thickness, h. Write a formula to represent this function. *(#13p.67)

Example: The average velocity, v, for a trip overa fixed distance, d, is inversely proportional to the time travel, t. Write a formula to represent this function. *(#15 p.67)

Example: Kleiber's Law states that the caloric needs of a mammal are proportional to its body weight raised to the 0.75 power. Surprisingly, the daily diets of mammals conform to this relation well. Assuming Kleiber's Law holds: *(#20 p. 68)

a) Give a formula for C, daily calorie needs, as a function of body weight, W.

b) If a human weighing 150 pounds needs to consume 1800 calories a day, estimate the daily calorie requirements of a horse weighing 700 pounds and a rabbit weighing 9 pounds.

Example:

* *Applied Calculus*, Hughes-Hallett et al, 4th Edition, Wiley, 2010.