

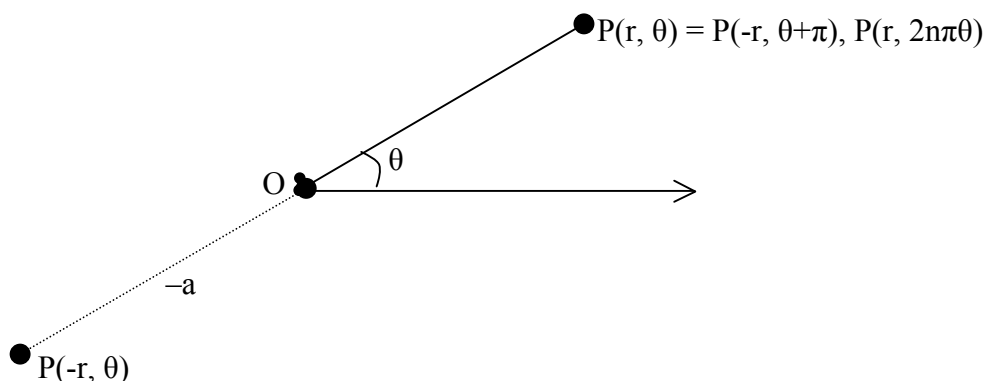
## §8.1 Polar Coordinates

**Polar coordinates** uses *distance* and *direction* to specify a location in a plane.

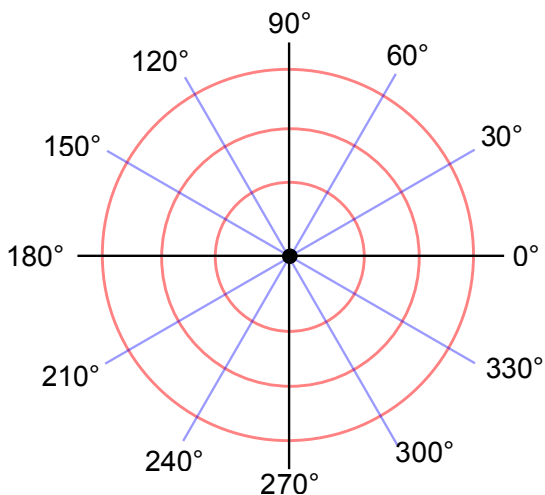
The **origin** in a polar system is a fixed point from which a ray,  $O$ , is drawn and we call the ray the **polar axis** (what we would have called the initial side in previous study). The distance from  $O$  to  $P$  ( $P$  is the endpoint of a ray rotated through an angle  $\theta$ ; what we would have called a terminal point in previous study). The distance from  $O$  to  $P$  is called “ $r$ ”, the radius.  $P$  is assigned an ordered pair determined by “ $r$ ” and  $\theta$ , and labeled with the **polar coordinates**  $P(r, \theta)$ .

In  $P(r, \theta)$   
 $r$  = distance from  $O$  to  $P$   
 $\theta$  = Angle that  $OP$  makes with the polar axis

*\*Note: A negative  $r$  lies on the opposite end of the ray with angle  $\theta$*



**Example:** Find 2 other polar coordinates that represent  $P(2, \pi/6)$  &  $P(-3, 2\pi/3)$  and graph them on the polar coordinate system shown.



The relationship between Rectangular & Polar coordinates is the relation that we saw in right triangles on the unit circle. The following is what we need to know to change **polar to rectangular coordinates**.

$$\frac{x}{r} = \cos \theta \quad \text{so} \quad x = r \cos \theta$$

$$\frac{y}{r} = \sin \theta \quad \text{so} \quad y = r \sin \theta$$

Thus,  $P(1, \pi/6)$  is  $\left. \begin{array}{l} x = 1 \cos \pi/6 = 1 \cdot \sqrt{3}/2 \\ Y = 1 \sin \pi/6 = 1 \cdot 1/2 \end{array} \right\}$  in the rectangular system

$P(2, \pi/6)$  is  $\left. \begin{array}{l} x = 1 \cos \pi/6 = 2 \cdot \sqrt{3}/2 = \sqrt{3} \\ Y = 1 \sin \pi/6 = 2 \cdot 1/2 = 1 \end{array} \right\}$  in the rectangular system

**Example:** Find  $P(-3, 2\pi/3)$  in the rectangular system

**\*Note:** We saw that this point lies in QIV earlier and that is what the rectangular coordinates show.

Now, we will go from **rectangular to polar coordinates**. We will need to know the following to help us in this process.

$$r^2 = x^2 + y^2 \quad \& \quad \tan \theta = \frac{y}{x}$$

**Example:** What are the polar coordinates of  $(-\sqrt{6}, -\sqrt{2})$   
 \*#38 p. 546 of Stewart's *Precalculus* Ed. 6

**\*Note:** You must be careful because the quadrant information must still conform, hence why we don't use  $\pi/6$  and instead use  $7\pi/6$

Our next task is to rewrite equations from their rectangular form to their polar form.

- 1) We will use some of the following techniques
  - a) Direct Substitution –  $x = r \cos \theta$  &  $y = r \sin \theta$
  - b) Algebra to manipulate
  - c) Squaring sides & or multiplying sides by the same constant
- 2) We will use Ch. 7 skills to simplify the resulting equation
- 3) Pay attention because the relation between polar & rectangular coordinates don't have unique solutions

**Example:** Convert the following equation to its equivalent polar equation

\*#46 p. 546 of Stewart's *Precalculus* Ed. 6

$$y = 5$$

Next, we will convert polar equations to rectangular equations.

- 1) Use the same techniques as above
  - a) Don't forget  $r^2 = x^2 + y^2$  &  $\tan \theta = y/x$
- 2) Use the identities in Ch. 7
- 3) Multiply by  $r$  or square sides or complete the square
- 4) Use techniques in 3 multiple times

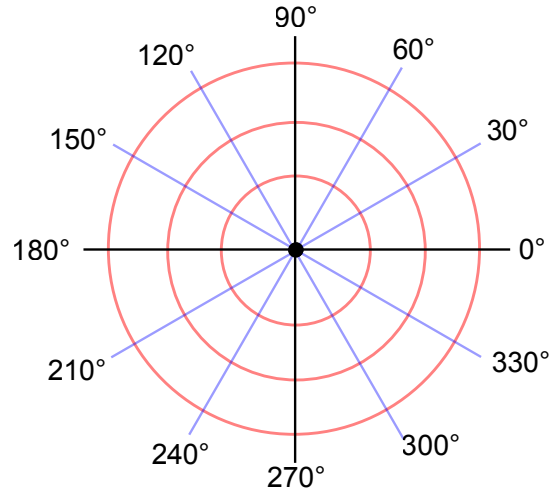
**Example:** Write the polar equations as equivalent rectangular equations

\*#60, 52 & 66 p. 546 of Stewart's *Precalculus* Ed. 6

a)  $r = 2 - \cos \theta$       b)  $\theta = \pi$       c)  $r^2 = \sin 2\theta$

## §8.2 Graphs of Polar Equations

The Polar Coordinate System consists of concentric circles with their center at O, the origin. The circles represent  $r$ , the lines emanating from O represent rays with varying angles  $\theta$  made with the polar axis. Here is one example of a polar system.



Some places that you might locate polar graph paper for your use are:

<http://www.marquis-soft.com/graphpapereng.htm>

[http://mathematicshelpcentral.com/graph\\_paper.htm](http://mathematicshelpcentral.com/graph_paper.htm)

Here's what you need to learn from this section – just as in Algebra, certain functions have certain graphic representations and certain features that lead to key graphic features. We are going to learn some of the basic polar equations and their graphic representations and the we will talk about their symmetry. Here is what you need to know, much of which you will find in your book on p.. 553 of Ed 6 & 594 of Ed 5:

Circles	$r = a$ (center at zero) $r = 2a \sin \theta$ (center at $ a , (a, \pi/2)$ ) $r = 2a \cos \theta$ (center at $(a, 0)$ )	
Spiral	$r = a\theta$	
Straight Lines thru Origin	$\theta = a$	
Cartoids	$r = a(1 \pm \cos \theta)$ $r = a(1 \pm \sin \theta)$	
Roses	$n$ -leaved when $n$ is odd $2n$ -leaved when $n$ is even         } $r = a \sin n\theta$ or $r = a \cos n\theta$	
Limacon	$r = a \pm b \cos \theta$ $r = a \pm b \sin \theta$	If $a < b$ then there is a loop, if $a > b$ then dimpled If $a = b$ then it is a Cartoid

Lemniscates	$r^2 = a^2 \sin 2\theta$	symmetric about $y = x$
	$r^2 = a^2 \cos 2\theta$	symmetric about the x-axis

You need to go over the examples on p. 548-551 of Ed 6 (p. 589-591 of Ed 5) paying attention to:

- 1) Rectangular Graph (if even possible)
- 2) One to one correspondence of the rectangular & polar coordinates in Example 1 through 6
- 3) Link between §8.1 & 8.2 in changing between polar & rectangular coordinates. Note that an easy change seems to show 1:1 correspondence in graphs.
- 4) Pay attention to the equations & picture lines as this is probably one of the places that I will test you.  
 Ex.  $r = 2 + 2 \cos \theta$       What would the general shape of the graph be?
- 5) Symmetry (we will discuss this next)

There are 3 types of **symmetry** in graphs of polar equations:

**1) Symmetry about the Polar Axis**

Test for it by:  $\theta$  vs  $-\theta$  giving the same value

**Ex.** Show that  $r = 1 + 2 \cos \theta$  is symmetric about the polar axis

*Note: This is easily shown in general by using the even/odd property for cosine*

**2) Symmetry about the Pole**

Test for it by:  $r$  vs  $-r$  giving the same value

**Ex.** Show that  $r^2 = 4 \cos 2\theta$  is symmetric about the pole

*Note: This is quite simply done by Algebra since  $r^2$  is same for  $\pm r$*

**3) Symmetry about the Vertical Line  $\theta = \pi/2$**

Test for it by:  $\theta$  vs  $\pi - \theta$  giving the same value

**Ex.** Show that  $r = \frac{4}{3 - \sin \theta}$  is symmetric about  $\theta = \pi/2$

*Note: This is done by difference of angles formula.*

Symmetry is another feature that can be used to identify a specific function from its graph.

Using our calculator to graph a polar equation:

- 1) Set the MODE to degrees & pol
- 2) WINDOW settings
  - $\theta$  step: smaller = more smoothness of curve
  - $\theta$  min &  $\theta$  max will tell you the number of revolutions & which ones you will see (0 to  $360^\circ$  is the 1<sup>st</sup> revolution)
  - x min & x max
  - y min & y max

**Ex.** Graph  $r = \theta \sin \theta$

- 1) This is an function that will have symmetry about the vertical line  $\pi/2$  because it involves the sine function
- 2) Start with a ZTrig setting and then fine tune from there
- 3) Don't forget to use negative  $\theta$  to investigate too

### §8.3 Polar Form of Complex Numbers & DeMoivre's Theorem

Recall the following about complex numbers:

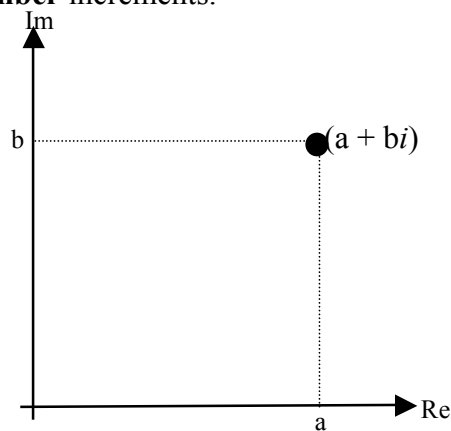
Complex numbers are numbers that have an imaginary component.

$$a + bi \quad a, b \in \mathfrak{R} \text{ and "i" is the imaginary unit}$$

Recall the imaginary unit  $i$ .

$$i^2 = -1 \text{ and therefore } \sqrt{-1} = i$$

Graphing a complex number is done in a system that looks like the rectangular coordinate system except the horizontal axis is the Real Axis (Re) – the  $a$ 's from a complex number & vertical axis is the Imaginary Axis (Im) – the  $bi$ 's from a complex number. Each axis is counted off using **real number** increments.



**Example:** Graph a)  $5 - 2i$       b)  $3 + \sqrt{2}i$       c)  $-1 + i$

Sets of complex numbers can be graphed too. This is similar to graphing linear inequalities in two variables such as  $x \geq 5$  &  $y < 3$ . Recall that the following is what such a graph would look like.

Now, we can use this same idea to graph a set of complex numbers.

**Example:** Graph  
a)  $\{a + bi \mid a \geq 5\}$                       b)  $\{a + bi \mid a \geq 5 \ \& \ b < 3\}$

The next tiny little concept we want to learn is the absolute value of an imaginary unit. This is called the **modulus (plural moduli)** of a complex number.

<p><b>Modulus of a Complex #</b></p> <p>For <math>z = a + bi</math></p> $ z  = \sqrt{a^2 + b^2}$ <p><math> z </math> is read as the modulus of <math>z</math></p>
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**Example:** Find the moduli for  
a)  $z = -5 + 3i$                       b)  $z = 4 - \sqrt{2}i$

We can graph the modulus of  $z$  too. When we **graph the modulus** we get a **circle of radius  $|z|$** .

**Example:** Graph  $\{z \mid |z| > 1/2\}$

Again, we want to relate this set of numbers to polar coordinates too. The following will be our key in doing this.



### Polar (Trig) Form of a Complex #

For  $z = a + bi$   
the polar form is:  $z = r (\cos \theta + i \sin \theta)$

where  $r = \text{modulus} = |z| = \sqrt{a^2 + b^2}$   
&  
 $\tan \theta = b/a$   
&  
 $\theta$  is called the **argument** of  $z$

**Example:** Find the polar form of  
a)  $-3 + \sqrt{3}i$       b)  $-\sqrt{2} - \sqrt{2}i$       c)  $-5 + 3i$

Multiplying and Dividing Complex numbers is another skill we will need. Most of the time you will want to multiply & divide using radians (occasionally it will be easier and more accurate to use degrees).

### Multiplying & Dividing Complex #

For  $z_1 = r (\cos \theta + i \sin \theta)$  &  $z_2 = r (\cos \theta + i \sin \theta)$   
which are the polar forms of 2 complex numbers

We **multiply** by multiplying the moduli & adding the arguments.  
 $z_1 \cdot z_2 = r_1 r_2 [\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]$

We **divide** by dividing the moduli & subtracting the arguments.  
 $\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]$

*Note: Use radians to operate in most cases.*

**Example:** For  $z_1 = 2\sqrt{3} (\cos 5\pi/6 + i \sin 5\pi/6)$  &  $z_2 = 2 (\cos 5\pi/4 + i \sin 5\pi/4)$   
a) Find  $z_1 \cdot z_2$       b) Find  $z_1 \div z_2$



Using DeMoivre's Theorem the following can be shown. It is with this "theorem" that we can in turn solve equations involving complex numbers.

**Nth ROOT of a Complex #**

For  $z = r (\cos \theta + i \sin \theta)$  the polar forms of a complex number

Where  $n = "+"$  integer representing "n" distinct roots of  $z$   
 $r =$  modulus of  $z$

The  $k$  roots can be found

$$w_k = r^{1/n} \cos \left[ \frac{(\theta + 2k\pi)}{n} + i \sin \frac{(\theta + 2k\pi)}{n} \right] \quad \text{where } k = 0, 1, 2, 3, \dots, n - 1$$

**Note:**  $\theta + 2k\pi$  is just saying all the **coterminal angles** (not reference angles)

e.g.  $\pi/3 = 2\pi + \pi/3 = 4\pi + \pi/3$

To get the  $n^{\text{th}}$  root you must consider **all** coterminal angles up to  $(n-1)2\pi$ 's  
 (which gives you all roots within a circle of radius  $r^{1/n}$ )

**Note 2:** Degrees may be easier to use in this case.

**Example:** Find the cube roots of  $4\sqrt{3} + 4i$

And now, you can use your skills to solve an equation using the  $n^{\text{th}}$  root.

- Example:** Solve & graph  $z^6 - 1 = 0$
- 1) Prepare to solve just as if it were  $x^2 - 1 = 0$
  - 2) Take the 6<sup>th</sup> root of both sides
  - 3) There will be  $k = 0, 1, 2, 3, 4, 5$  roots
    - a) Get polar form of  $z$  from step 2
      - i) Graph for easier  $\theta$  interpretation
      - ii) Modulus
      - iii)  $\theta$
      - iv) Give polar form of  $z$
    - b) Use the above theorem to give the 6 roots
    - c) Graph them on a circle of radius  $r^{1/n} = 1$

## **§8.4 Plane Curves & Parametric Equations (§10.7 in Ed 5)**

**Plane curves** show a point moving in a plane. In other words a plane curve has ordered pairs as well as a direction of movement as  $t$ , time, increases.

### **Sketching a Plane Curve**

- 1) Find  $x$  &  $y$  coordinates for parametric equations over an interval containing both positive & negative values of  $t$ .
- 2) Plot  $(x, y)$ 's in the rectangular coordinate system
- 3) Draw arrows to represent the direction of movement as  $t$  increased.

**Example:** Sketch  $x = 2t + 1$        $y = (t + 1/2)^2$   
#4a p. 807 from Ed. 5 of Stewart's Precalculus

*Note: I used  $-7/2 \leq t \leq 5/2$  by increments of  $1/2$ 's.*

Next, we want to take an equation from its parametric form to its equivalent rectangular form by **eliminating a parameter**.

### **Steps for Writing a Parametric Equation as a Linear Equation**

- 1) Solve the simpler equation for the parameter  $t$
- 2) Substitute into the other equation
- 3) Simplify

**Example:** Give the linear form for  $x = 2t + 1$  &  $y = (t + 1/2)^2$

*Note: The equation we obtain is a parabola and notice how the points that we obtained before conform to the equation.*

What if the equations are trigonometric in type? Then we will follow the following procedures

### Steps for Parametric Equations as Trig Functions

- 1) Find an “identity” that relates the functions
- 2) Substitute the rectangular coordinate values in for the trig function in step 1
- 3) Simplify
- 4) Make sure the range of the trig function(s) are notes (a graph of the parametric can help here)

**Example:** Give the linear form of  $x = \cot t$  &  $y = \csc t$   
\*#18 p. 807 Stewart’s Precalculus Ed. 5 on  $0 < t < \pi$

### Creating the Parametric Equation for a Line

- 1) Know or find the slope of the line
- 2)  $x = x\text{-coordinate} + t$  (Use the lowest known ordered pair)
- 3)  $y = y\text{-coordinate} + mt$  (Use lowest known again, from x’s ordered pair)

**Example:** Find the parametric equation for a line thru (12, 7) & the origin  
\*#26 p. 807 Stewart’s Precalculus Ed. 5

There are two types of curves described by parametric equations that are worth note:

**Cycloid** The position of a point P on a circle of radius, r, as it is rolled along the x-axis.

$$x = a(\theta - \sin \theta) \text{ \& \ } y = a(1 - \cos \theta)$$

Where a = radius &  $\theta$  = angle of the arc subtended in the roll

**Lissajous Figure** These are “figure 8” type figures described by  

$$x = A \sin \omega_1 t \quad y = B \cos \omega_2 t$$
A & B can be used to find the width & height of the figure  
 $(-A \leq x \leq A \text{ \& } -B \leq y \leq B)$  and  $\omega_1$  tells us how many loops  
vertically &  $\omega_2$  tells us how many horizontal loops.

Let’s try graphing a Lissajous Figure with our calculator

- Example:** On your TI graph  $x = \sin 4t$  &  $y = \cos t$   
\*#40 p. 807 Stewart’s Precalculus Ed. 5
- Step 1: MODE & set to radians & PAR
- Step 2: Y =  $X_{1T} = \sin(4t)$  t using xTθn key  
 $Y_{1T} = \cos(t)$
- Step 3: WINDOW Leave t min, tmax & tstep alone  
Set x min = -1, x max = 1, x scl = 0.1  
y min = -1, y max = 1, x scl = 0.1
- Step 4: GRAPH

Now alternate, make  $y = \cos 3t$  while  $x = \sin t$  and then try to do both together with  $\omega$ ’s.

Our last task is to express a Polar Equation in Parametric Form

**Parametric Form of a Polar Eq.**

If  $r = f(\theta)$  is in polar form, then

$$x = f(t) \cos t \quad \& \quad y = f(t) \sin t$$

Is the parametric form of the polar equation

- Example:** Express  $r = \sin \theta + 2 \cos \theta$  in parametric form & graph  
\*#44 p. 807 Stewart’s Precalculus Ed. 5