§7.1 Trigonometric Identities

We will again be studying some of the same identities discussed in Ch. 5 & 6.

Reciprocal Identities

$\tan t = \dfrac{\sin t}{\cos t}$

$\cot t = \dfrac{1}{\tan t} = \dfrac{\cos t}{\sin t}$

$\sec t = \dfrac{1}{\cos t}$

$\csc t = \dfrac{1}{\sin t}$

Pythagorean Identities

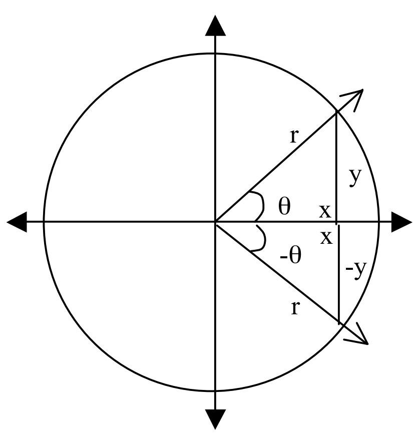
$\sin^2\theta + \cos^2\theta = 1$ or $\cos^2\theta = 1 - \sin^2\theta$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

And, we will be studying some new identities that I have mentioned in my discussions of topics in Ch. 5 & 6.

First, I will give you a brief synopsis of the Even-Odd Identities & then I will summarize them.



$\sin(\theta) = \dfrac{y}{r}$ and $\sin(-\theta) = \dfrac{-y}{r}$

so,

$\mathbf{\sin(-\theta) = -\sin(\theta)}$

and

$\cos(\theta) = \dfrac{x}{r}$ and $\cos(-\theta) = \dfrac{x}{r}$

so,

$\mathbf{\cos(-\theta) = \cos(\theta)}$

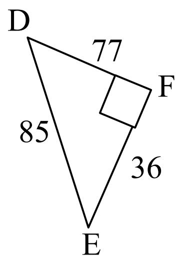
And,

$$\mathbf{Tan(-\theta)} = \frac{\sin(-\theta)}{\cos(-\theta)} = \frac{-\sin(\theta)}{\cos(\theta)} = -\frac{\sin(\theta)}{\cos(\theta)} = \mathbf{-\tan(\theta)}$$

Note: *Remember that in§ 5.2 we learned sine is an odd f(n) [it's symmetric about the origin, so if (x, y), then (-x, -y)] and cosine is an even f(n) [it's symmetric about the y-axis so if (x, y) then (-x, y)]*

The other identities that we have discussed, but not fully defined are the cofunction identities. We will investigate these by example (that is how it came up in our discussion earlier btw).

Example: Find the 6 trigonometric functions for ∠'s D & E in the figure to the right and fill in the table on the left.



∠D	∠E
sin D	sin E
cos D	cos E
tan D	**tan E**
cot D	**cot E**
sec D	sec E
csc D	csc E

***Example:** For the last example what, what do you notice about sin D & cos E or cos D & sin E, tan D & cot E or cot D & tan E, sec D & csc E or csc D & sec E?

What you have just observed are **<u>cofunctions</u>**. If we draw the triangle we see above we see that opposite ∠D is adjacent ∠E, so sin D = cos E and vice versa. This is also true for tan and cot since tan = opp/adj. tan D = cot E. Since this is true of sin and cos, we can show that it holds for the reciprocals of sin and cos – csc and sec (respectively), hence csc D = sec E. Since the 2 angles that aren't the right ∠ in a right Δ are complementary (∠'s sum to 180°, so if one is 90° the other 2 must sum to 90°), this combined with the idea of cofunctions leads to the **<u>6 Cofunction Identities</u>**.

Even-Odd Identities

sin (-θ) = - sin (θ)

cos (-θ) = cos (θ)

Tan (-θ) = - tan (θ)

Cofunction Identities

sin A = cos (90 – A)° **or** cos A = sin (90 – A)°

tan A = cot (90 – A)° **or** cot A = (90 – A)°

sec A = csc (90 – A)° **or** csc A = sec (90 – A)°

*Note: 90 = $^{\pi}/_{2}$ which is what your book uses

We have been doing a little of this manipulation since Chapter 5. Let's go over some general guidelines that your book uses to help you to **prove an identity**. Keep in mind that this is a skill that you will be applying in Calculus.

<u>Guidelines as Outlines by Stewart, Redlin & Watson (p. 530</u> <u>ed 5 & p. ed 6)</u>

1) Start with one side (usually the more complicated)
   a) Pick one side & write it down
   b) Goal is to make it into the other side using identities
2) Use known identitites
   a) Use algebra
      i) Recall a perfect square trinomial: $a^2 \pm 2ab + b^2$ factors as $(a \pm b)^2$
         Ex. $9x^2 + 18x + 1 = (3x + 1)^2$
         and
         $4x^2 - 20x + 25 = (2x - 5)^2$
      ii) Finding an LCD & Building Higher Terms
         Ex. $^1/_x + {}^1/_y = {}^y/_{xy} + {}^x/_{xy} = {}^{(x+y)}/_{xy}$
      iii) Multiplying Conjugates yields the difference of squares
         Ex. $(2 + x)(2 - x) = 4 - x^2$
   b) Select identities that will bring your complicated side to a more simplistic end (headed toward the desired outcome)
3) Convert to sines & cosines
   a) Use reciprocal & quotient identities when possible
   b) Resort to Pythagorean identities when necessary

***Note:** Your book warns of trying to transform something that is not equivalent at face value by doing the same thing to both sides. This is a skill that you are familiar with and may lapse into out of habit. Be forewarned! The book's example was sin x = -sin x proven by squaring both sides. If a process is not invertible it should not be used – remember that squaring is not invertible unless the domain is restricted.*

Sometimes it will be necessary to re-express one function in terms of another (we've already been doing this). This can be useful in graphing and in differentiation and integration in Calculus. To do this try the following:

1) Find an identity that relates the functions
2) Solve for the function of desire

Example: Express tan θ in terms of cos θ

Now, we'll do a little more of the same, but try to use only sin and cos. This is an important skill to possess, hence while we'll practice this skill by itself in this section. Here's a strategy:

1) Use sin & cos to re-write the function(s)
2) Simplify using algebra skills

Example: Write (sec θ + csc θ) in terms of sine and cosine and simplify

How about the use of a co-functions to re-write

Example: Write cos $(90 + \theta)$ as a single trig function of θ

Hint: Use the fact that $-(-\theta)$ is $+\theta$ and then use even/odd

Your Turn

***Example:** Write cos x in terms of csc x. (Hint: Think Pythagorean Id. & reciprocate)

***Example:** Write in terms of cosine and sine and simplify so no quotients appear. The final answer doesn't have to have sine and cosine.

Now we will try some more complicated "proofs".

Example: Prove that (cot x)(sin x)(sec x) = 1.

Hint: Put the more complex side in terms of sin & cos

Example: Prove that $\cot^2 \theta(\tan^2 + 1) = \csc^2 \theta$ is an identity.

Hint: Use Pythagorean id & rewrite in terms of sin & cos

Example: Prove that $\dfrac{\tan^2 x}{\sec^2 x} = (1 + \cos x)(1 - \cos x)$

Hint: Focus on right side using algebra to rewrite, restate cos in terms of sec and use algebra again to add fractions finishing with a Pythagorean

This next one we'll do by introducing "a little something extra" as your book says:

Example: Prove that $\dfrac{\sec x + \tan s}{\sin x} = \dfrac{\csc x}{\sec x - \tan x}$ is an identity

Hint: Focus on right. Use conjugate with fundamental theorem of fractions. Simplify & use another Pythagorean ID

Now let's practice using a different technique. Instead of focusing on just one side we'll **manipulate both sides**. We'll start with a second look at the third example above.

Example: Prove that $\dfrac{\tan^2 s}{\sec^2 s} = (1 + \cos s)(1 - \cos s)$

Hint: Rewrite left using sin & cos. Rewrite right by mult. out & using Pythagorean Id.

***Note:** *Now you have seen this problem worked in 2 ways. There is actually a 3rd way to work it as well. Focus on the left side & restate the sec using a Pythagorean identity restate the tan then multiply and factor.*

Example: Prove that $\dfrac{\cot\theta - \csc\theta}{\cot\theta + \csc\theta} = \dfrac{1 - 2\cos\theta + \cos^2\theta}{-\sin^2\theta}$

Hint: Right: Factor, Pythagorean Id, Factor
Left: Since csc & cot have sin in common, so mult by sin & simplify

One last skill that is needed in Calculus is a substitution skill. When on the unit circle recall that (cos x, sin x) are the coordinates of a point and r = 1, so since the Pythagorean Identity can be re-written as $\cos\theta = \sqrt{1 - \sin\theta}$ this means that $x = \sqrt{1 - y}$ and likewise since $\sin\theta = \sqrt{1 - \cos\theta}$ it means that $y = \sqrt{1 - x}$. Our book uses substitution to get you to think along these lines. Problems like this are at the end of the exercise set #89-94 in Ed 5 & #91-96 in Ed 6.

Let's do the following for practice:

Example: Make the substitution and rewrite. Assume $0 \le \theta \le \pi/2$

(*#90 & #94 p. 534 *Precalculus*, 5th ed, Stewart)

a) $\sqrt{1 + x^2}$ Where $x = \tan\theta$ b) $\dfrac{1}{x^2\sqrt{4 + x^2}}$ $x = 2\tan\theta$

§7.2 Addition & Subtraction Formulas

I debated whether or not to cover the derivation of these formulas in class. I have decided against it due to time constraints. Suffice it to say that the proof comes from

1) 2 points on a circle with terminal points described by the sine and cosine
2) The distance formula
3) Some algebra

I will leave it to a diligent student to investigate further on p. 535 of Ed. 5 & p. 500 of Ed 6.

Sine of a Sum/Difference	**Cosine of a Sum/Difference**
$\sin(A + B) = \sin A \cos B + \cos A \sin B$	$\cos(A + B) = \cos A \cos B - \sin A \sin B$
$\sin(A - B) = \sin A \cos B - \cos A \sin B$	$\cos(A - B) = \cos A \cos B + \sin A \sin B$

Tangent of a Sum/Difference

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

There are multiple tasks that we will need to perform using these sum/difference identities. Some of the tasks may have characteristics from our first section.

1) We may be required to rewrite an angle as a sum of 2 angles for which we know exact trig values.
   Ex. $15^\circ = 45^\circ - 30^\circ$ & $\frac{13\pi}{12} = \frac{9\pi}{12} + \frac{4\pi}{12} = \frac{3\pi}{4} + \frac{\pi}{3}$

Note: *Here are some equivalencies that you might find helpful in doing these problems.*
$\frac{2\pi}{12} = \frac{\pi}{6} = 30^\circ$ $\frac{3\pi}{12} = \frac{\pi}{4} = 45^\circ$ $\frac{4\pi}{12} = \frac{\pi}{3} = 60^\circ$

2) We could be called to remember reference angles
   Ex. See the second example above
3) We could be required to use even/odd identities
   Ex. $\sin(-15^\circ) = -\sin 15^\circ$
4) We could use the identities in reverse with or without skills in 1-3 above
   Ex. $\dfrac{\tan 100^\circ - \tan 70^\circ}{1 + \tan 100^\circ \tan 70^\circ} = \tan(100^\circ - 70^\circ)$
5) We might use the identities in general resulting in the use of exact trig values simplifying down to a single trig function in general
   Ex. Prove $\sin(\theta - 270^\circ) = \cos\theta$
6) We may use a special formula that comes from substitution in the last case to re-express a sum so that it is easier to graph as a translation of sine
   *See below for the formula & substitution

Example: Find the exact value for

a) $\cos(-75°)$ b) $\cos\left({}^{17\pi}/_{12}\right)$ c) $\cos 173° \cos 83° + \sin 173° \sin 83°$

d) $\sin(-15°)$ e) $\tan\left({}^{13\pi}/_{12}\right)$ f) $\dfrac{\tan 100° - \tan 30°}{1 + \tan 100° \tan 70°}$

Example: Prove that $\cos(180 + \theta) = -\cos\theta$ using the sum/difference identities.

The last example is a part of a group of formulas known as the **reduction formulas**. They are not covered in your text at this time, but since problems #21-30 in Ed 6 (#19-27 in ed 5) make you prove many of them I thought I would include them here.

Reduction Formulas

$\cos(90 + \theta) = -\sin\theta$	$\cos(180 - \theta) = -\cos\theta$
$\cos(180 + \theta) = -\cos\theta$	$\cos(270 - \theta) = -\sin\theta$
$\sin(180 + \theta) = -\sin\theta$	$\tan(270 - \theta) = \cot\theta$

Example: Prove each of the following

a) $\sin(\theta - 270°) = \cos\theta$ b) $\tan(\theta + 3\pi) = \tan\theta$

Your book makes a point to give you the following theorem.

Sums of Sines and Cosines

If A & B ∈ Real #'s then

$$A \sin x + B \cos x = k \sin(x + \phi)$$

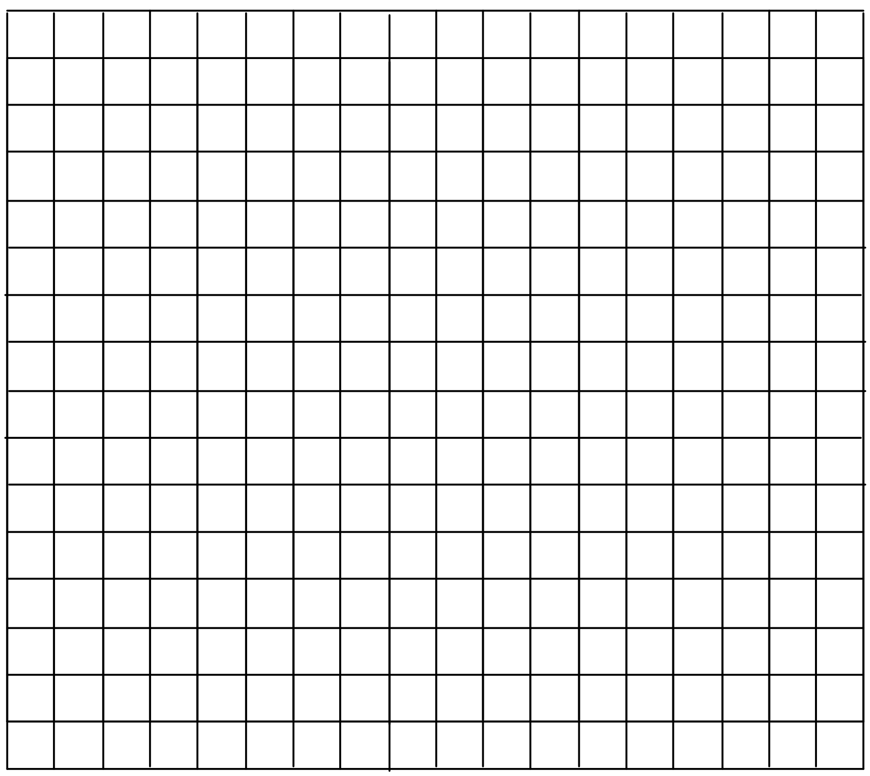
Where $k = \sqrt{A^2 + B^2}$ and ϕ satisfies

$$\cos\phi = \frac{A}{k} \quad \& \quad \sin\phi = \frac{B}{k}$$

Example: Prove that ${}^{\sqrt{3}}/_{2} \cos\theta + -{}^{1}/_{2} \sin\theta = \sin(120^\circ + \theta)$

Example: Use what you've just learned to rewrite & then graph

$$f(x) = \sin x + \cos x$$



§7.3 Double-Angle, Half-Angle & Product-Sum Formulas

Double Angle Identities

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 2A = 1 - 2\sin^2 A$

$\cos 2A = 2\cos^2 A - 1$

$\sin 2A = 2\sin A \cos A$

$\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

Half Angle Identities

$\cos \dfrac{A}{2} = \pm\sqrt{\dfrac{1 + \cos A}{2}}$

$\sin \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{2}}$

$\tan \dfrac{A}{2} = \pm\sqrt{\dfrac{1 - \cos A}{1 + \cos A}}$

$\tan \dfrac{A}{2} = \dfrac{\sin A}{1 + \cos A}$

$\tan \dfrac{A}{2} = \dfrac{1 - \cos A}{\sin A}$

* + or – depends on quadrant of $^{A}/_{2}$

Sum to Product

$\sin A + \sin B = 2\sin\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

$\sin A - \sin B = 2\cos\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

$\cos A + \cos B = 2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

$\cos A - \cos B = -2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{A-B}{2}\right)$

Product to Sum

$\cos A \cos B = {}^{1}/_{2}\,[\cos(A+B) + \cos(A-B)]$

$\sin A \sin B = {}^{1}/_{2}\,[\cos(A-B) - \cos(A+B)]$

$\sin A \cos B = {}^{1}/_{2}\,[\sin(A+B) + \sin(A-B)]$

$\cos A \sin B = {}^{1}/_{2}\,[\sin(A+B) - \sin(A-B)]$

Formulas to Lower Powers

$\sin^2 A = \dfrac{1 - \cos 2A}{2}$ $\qquad \cos^2 A = \dfrac{1 + \cos 2A}{2}$ $\qquad \tan^2 A = \dfrac{1 - \cos 2A}{1 + \cos 2A}$

*Note: These are just the double angle formulas from above manipulated. Some books don't even separate them out.

One of the ways that we will see the **DOUBLE ANGLE & HALF ANGLE** formulas used is to use our basic knowledge of the 45/45/90 and 30/60/90 triangles to angles that are half or twice those values.

Example: Practice your skills with the double angle formula by rewriting the angle to see it as twice a known angle. You probably already know all the exact values anyway, but that is not the point.

a) $\cos 90^{\circ}$ b) $\cos 120^{\circ}$ c) $\cos {}^{\pi}/_{3}$

d) $\sin {}^{\pi}/_{2}$ e) $\tan -60^{\circ}$

Example: Find the exact value of $\sin 22.5^{\circ}$ by using your skill with the 45/45/90 triangle.

Example: Given that $\sin\theta = {}^{8}/_{17}$ and $\cos\theta < 0$, find $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$

Example: Given $\cos x = -{}^{3}/_{7}$ with $\pi < x < {}^{3\pi}/_{2}$ find $\sin({}^{x}/_{2})$, $\cos({}^{x}/_{2})$, $\tan({}^{x}/_{2})$

Example: Use the double angle formula to find all the values of the 6 trig f(n) of θ given that $\cos 2\theta = -{}^{12}/_{13}$ and $180^\circ < \theta < 270^\circ$

Example: Using a half-angle formula rewrite: $\sqrt{\dfrac{1 - \cos 8x}{2}}$

Note: *This is very simple. A = argument of the cosine so see the 1/2 angle formula and use it with* ${}^{A}/_{2}$.

Example: Verify that the following is an identity: $\cos^4 x - \sin^4 x = \cos 2x$

Hint: *You'll have to use difference of squares & an identity as well as a double angle formula.*

Next we can do some problems that come from the **product to sum** or **sum to product**. The product to sum identities are simple to find by adding the sum and difference formulas from our last section.

Example: Write $6 \sin 40^\circ \sin 15^\circ$ as a sum/difference of functions.

Example: Write $\cos 3\theta + \cos 7\theta$ as a product of two functions.

Here's one that uses a few of the formulas together.

Example: Write cos 3x in terms of cos x.

Hint: *Use a sum to rewrite 3x and then manipulate that using a formula, finally wrapping up with some double angle formulas and some simplification.*

§7.4 Solving Basic Trig Equations

Recall conditional linear equations in Algebra had 1 solution. We are going to study conditional trig equations first. This means that some values will solve them and some will not.

Method 1: Linear Methods

1) Employ algebra to isolate the trig function
2) Use trig identities as needed
3) Use definitions as needed
4) When using the inverse to solve, make sure that you realize that the answer you get comes from the range of the inverse function & you may need to use that as a reference angle(point) to find all possible values for original equation.

Recall:	$\sin(\sin^{-1} x)$	on $[-1, 1]$
	$\sin^{-1}(\sin x)$	on $[-\pi/2, \pi/2]$
	$\cos(\cos^{-1} x)$	on $[-1, 1]$
	$\cos^{-1}(\cos x)$	on $[0, \pi]$
	$\tan(\tan^{-1} x)$	on $(-\infty, \infty)$
	$\tan^{-1}(\tan x)$	on $(-\pi/2, \pi/2)$

5) Also realize that when finding all possible solutions that each of the functions repeats with a certain period and once the solutions are found in the fundamental period they can then be expanded to encompass all periods.

Recall: sine & cosine's period is 2π

$\therefore$ 2 values w/in fundamental cycle $+ 2k\pi$
where **k** is an integer

tangent's period is π

$\therefore$ 2 values w/in fundamental cycle $+ k\pi$
where **k** is an integer

Example: Solve $3 \tan \theta - \sqrt{3} = 0$ on $[0, 360^\circ)$

Method 2: Factoring & Quadratic Methods

1) Get f(x) = 0 using algebra
2) Factor & use zero factor property
3) Use Method 1's process if needed

Example: Solve $\tan x \sin x + \sin x = 0$

Example: Solve $\cos \theta \cot \theta = -\cos \theta$ on $[0, 360^\circ)$

<u>§7.5 More Solving Trig Functions</u>

This is really just a continuation of the last section. We just get more methods for solving and have the added problem that sometimes we will have **extraneous roots** that we will need to catch as we did when solving rational equations in algebra.

<u>Method 3: Using Trig Identities</u>

1) See a relationship that exists between 2 functions
2) Square sides in order to bring out an identity – be careful to check solutions for extraneous roots any time you do this
3) Use an identity to rewrite in terms of one function
4) Solve as in the last section

Example: Solve $\cot x - \sqrt{3} = \csc x$ on $[0, 2\pi)$

Note: *csc and cot are related via a Pythagorean Identity. You can create a problem by squaring to get there, so do check extraneous roots.*

Example: Solve $\cos 2x = \sin x$ on $[0, 2\pi)$

Note: *Did you get 3 solutions? Don't forget about reference angles.*

Example: Solve $2\cos^2\theta - 2\sin^2\theta + 1 = 0$ on $[0, 360^{o})$

Note: *Because we end up solving for 2A, we need to make sure we answer for A by dividing all by 2!*

Example: Solve $\cos 2x + \cos x = 0$