Chapter 8: Elementary Probability Theory

§8.1 Sample Space, Events & Probability

All of the counting techniques and the ways of listing events that we discussed in Ch. 7 will now be used in defining probabilities here in chapter 8. First we must talk about the vocabulary that we use in probability. We must have in mind a **sample space** (a universe), define an **event** (a subset of the universe/a subset of the sample space) and then be able to talk about a **simple event** (an element of a subset of the universe; namely the sample space).

A **sample space** is all possible outcomes (simple events) for an experiment. We usually denote a sample space with a capital S or S.

Example: All the ordered pairs that result from the roll of a pair of dice yield a sample space.

$$S = \{(x,y) \mid x,y \in 1,2,3,4,5,6\}$$

Example: The possible outcomes of the flip of a coin.

$$S = \{\text{heads, tails}\}$$

Your Turn: Give the sample space for the possible outcomes for the flip of 3 coins (flipping a single coin 3 times).

***Note:** A perfect method of finding all the possible combinations is to draw a tree diagram where each level represents a coin and each branch represents one of the two possible outcomes.

These sample spaces come about in accordance with an **experiment** which we already know is a process by which we gather information. Because experiments may be looking for something in particular (the population) sample spaces must be defined accordingly.

An **event** is an outcome or collection of outcomes from an experiment. It is equivalent to a set defined within the universe in set theory (in other words a subset of S).

Example A: The sum of the pair of dice is an event.

A = {2, 3, 4, 5, 6, …, 12} describes this event

Example B: The ordered pair that represents a sum of seven on a pair of dice when 2 die are rolled simultaneously.

B = {(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

Your Turn: Describe the event of getting 2 heads when flipping 3 coins simultaneously?

A simple event is any element of the sample space for which there is no further break down. For instance:

Example: In Example A above, 2 can be comprised of (1,1) which is the simple event

In Example B we are already referring to a set of simple events since you can't break down the elements within this set any further (you can't break down to the first die # or the second die number because the sample space is comprised of the rolls from 2 dice). As a result (1,6) is a simple event.

In your example you wrote down the set of simple events. Therefore a simple event could have been HH or HT.

Now we're able to talk about **probabilities**. We talk about the probabilitiy of an event occurring once a sample space has been thoroughly defined. A probability is the likelihood of occurrence of an event.

Probability is the likelihood of the occurrence of an event. It is a numeric measure of this likelihood that must be a number between 0 and 1. I'm going to talk about the "heart-and-soul" of probability before we continue.

Heart-and-Soul Rules of Probability

1) All probabilities must be between zero and one

$$0 \leq P(A) \leq 1$$

2) If the event can't possibly happen then there is zero probability.

$$P(A) = 0$$

3) If there is only one way in which an event can occur (the event is equivalent to the entire sample space) then the probability is one.

$$P(A) = 1$$

There are 3 ways of defining probability:

1) The **intuitive** method for finding the probability of an event.

   Example: The probability that another planet, like the Earth, could exist in the universe in the next 4 million years, according to Prof. Watson is less than 0.0001

2) The **relative frequency/empirical** approach for finding the probability of event A.

   P(A) = f / n where f = frequency of occurrence **and**
   n = total number of outcomes

   Example: After flipping a coin 15 times, the following outcomes were noted: H, H, T, T, T, T, T, H, T, T, H, T, H, T
   P(H) = 5/15 = 0.33

Note: *This data was created using EXCEL. I used* =ROUND(RAND()*(1-0)+0,0) to create 15 random digits and then I coded them using the following =IF(A1=0,"H",IF(A1=1,"T"))
Note2: *Probabilities are usually rounded to 2 significant digits (meaning 2 actual digits, not place values)*

3) The **classic /theoretical approach**, based upon equally likely* outcomes.

   P(A) = s / n where s = number of ways event A can occur **and**
   n = number of simple events

   Example: What is the probability of drawing a king from an unmarked deck of 52 card?

   First: Define the number of simple events (there are two ways to do this for this particular problem; we'll be using the most basic approach – the cards).
   Second: How many ways can you get a king
   Finally: P(K) = $^4/_{52} = {}^1/_{13}$

Note: *A probability can be expressed as a decimal or as a fraction in lowest terms or as a percentage.*
**Note*: Your book refers to this as the equally likely assumption.*

Counting techniques and theoretical probabilities. The **equally likely assumption** is the first and foremost assumption in each case. Here is the process that you must undergo in finding a theoretical probability regardless of the counting technique being applied.

Step 1: Equally likely assumption P(E) = n(E) ÷ n(S)
S is the sample space
every simple event in the sample space S is equally likely to occur
where E is the event
Step 2: Use a counting technique to find n(s)
Step 3: Define E & use a counting technique (if necessary) or a diagram to find n(E)
Step 4: Find P(E) by dividing n(E) by n(S)

Next are some examples where no fancy counting techniques are required to find probabilities. Being a visual person myself I always find these quite simple using the sample space or a tree diagram, but I will include counting technique information below each example.

Example: In a bag, I have the numbers 1 through 18 written on slips of paper. If I reach in and grab a slip of paper, what is the probability that it is a number divisible by 12?

***Note**: No counting technique is necessary here. This is simply a matter of listing the elements in the sample space and defining the event.

Example: In a bag, I have the numbers 1 through 18 written on slips of paper. If I reach in and grab a slip of paper, what is the probability that it is a number greater than 15?

***Note**: No counting technique is necessary here. This is simply a matter of listing the elements in the sample space and defining the event.

Example: We toss 3 coins simultaneously (or separately it doesn't matter) and we are interested in the probability that there are 3 heads? Draw the tree diagram to help with this problem.

***Note**: The counting technique employed here is for n(S). It uses the multiplication rule. There are 2 outcomes for each toss and there are 3 tosses so the total number is 2^3.
***Note2:** This type of probability can also be found using Binomial Theory, which we will not be covering. It is in your book however if you wish further investigation (see Ch. 11).

Example: We toss 3 coins simultaneously (or separately it doesn't matter) and we are interested in the probability that there are 2 heads and 1 tail, in any order? Draw the tree diagram to help with this problem.

***Note**: The counting technique employed here is for n(S). It uses the multiplication rule. There are 2 outcomes for each toss and there are 3 tosses so the total number is 2^3. The counting technique used for n(E) is a combination because you are counting the number of ways that among 3 things 1 can be of type 1 regardless of order (or likewise you can think of it as among 3 things 2 can be of type 1; the other has no choice but to be of the other type (binomial) and therefore it has choice of 1 (you would multiply the choices $C_{n,r}$ by 1)

Example: We toss 3 coins simultaneously (or separately it doesn't matter) and we are interested in the probability that there are 2 heads and 1 tail, in that order? Draw the tree diagram to help with this problem.

***Note**: The counting technique employed here is for n(S). It uses the multiplication rule. There are 2 outcomes for each toss and there are 3 tosses so the total number is 2^3. The counting technique used for n(E) is the arrangements because you are counting the number of ways that among 3 things 3 things can be chosen, giving the number of choices to be $P_{n,r}$

Example: Given the roll of 2 dice, what is the probability that the sum of 5 turns up?

***Note**: The counting technique employed here is for n(S). It uses the multiplication rule. There are 6 outcomes for each dice rolled and there are 2 dice to roll so the total number is 6^2. The n(E) requires the use of a new sample space, but can not be found using techniques explored thus far.

Example: A combination lock has a 5 digit code with no repeated digits, what is the probability of opening the lock if the choices of digits are 0 to 9? Draw the digits out as ___ ____ ____ ____ ____ and figure out how many possibilities there are based on the digit choices.

Note: The counting technique applied here is for n(S). We have a sequence (order matters) of 5 to be chosen from 10. Therefore n(S) is $P_{10,5}$. Only one of these can be our combination.

Now, we need to move away from problems where we can easily visualize all possible outcomes using tree diagrams, tables or sample spaces.

Example: Given a standard deck of 52 cards, what is the probability that a hand of 5 card stud will be a flush (all the same suit) in the suit of diamonds?

Note: The n(S) is all possible hands, which is all the possible ways to draw 5 cards from a 52 card deck – $C_{52,5}$. The n(E) is the number of possible ways to draw 5 cards from the suit of diamonds – which is $C_{13,5}$.

Example: The 4-H club consists of 16 girls and 12 boys. If 6 are randomly chosen to go a state 4-H event as representatives, what is the probability that 4 boys and 2 girls will be chosen to go?

Note: The n(S) is all possible 6 member groups from the entire club – $C_{28,6}$. The n(E) is a compound event, consisting of choices of 4 boys and 2 girls from their respective representing portions of the group. In other words it is $C_{16,2}$ multiplied by $C_{12,4}$.

Example: If 9 invitations are written and addressed to 9 people. The person that distributes the invitations did not know that they were addressed to specific people, and just handed them out. What is the probability that all of the invitations went to the right people?

Note: The key here is the number of ways the invitations could be distributed. This is the number of permutations of 9 things taken 9 at a time – $P_{9,9}$ or 9!.

§8.2 Union, Intersection & Complement Events: Unions

This section discusses the compound event in which two or more simple events occur. This is known as the union of 2 or more events. We discussed unions in Ch. 7 and discussed the fact that we will hear the word "or" used in association with these problems. The compound events' probability can be found using the **Addition Rule** of Ch. 7, which your book re-names here as Theorem 1: The Probability of the Union of 2 Events.

Probability of the Union of 2 Events (Addition Rule)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Recall from set theory that this means that A or B or Both occur.

If the events are mutually exclusive this rule becomes

$$P(A \cup B) = P(A) + P(B)$$

since there is no overlap of A & B

In conjunction with the compound event this section also discusses the complement of an event. The **complement** of an event and the event itself together comprise the entire sample space. Following are notations used by your book and others to represent the complement of an event, A.

A'	to is used by your book to denote the complement
$\overline{A}$	typically denotes the complement
A^c	some books even use this notation

Therefore, in terms of probability:

$$P(A') = 1 - P(A),$$

which can really help us in defining probabilities.

Example: If a coin is flipped 1000 times and 613 heads appeared, what is the probability of getting a tail?

A = Getting a head

$\therefore$ A' = Getting a tail since $A \cup A' = S$

$P(A) = 613/1000 = 0.613$

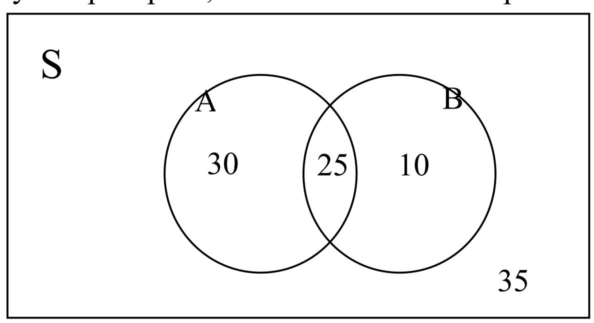
$P(A') = 1 - P(A) = 1 - 0.613 = 0.487$

Your Turn: Ten thousand people are surveyed and 2500 are found to like the product in question. What is the probability that a person does not like the product? Use the complement to find this probability.

There are two ways of visualizing the probability of a compound event:

1) Venn Diagrams
2) Tables (Two-Way or Contingency)

Example: The Venn Diagram below shows events A & B in an equally likely sample space, S. Find the indicated probabilities.



a) $P(A \cup B)$ b) $P[(A \cap B)']$

c) $P(A' \cap B)$ d) $P(A \cup B')$

d) $P[(A \cup B)']$ e) $P(A' \cap B')$

***Note:** d) & e) are the same exact things! You should make note of this for future reference.

Example : The probability that a randomly chosen family will own a color TV is 0.86, a black and white set is 0.35 and both types is 0.29. What is the probability that a randomly chosen family will own either a color or a black and white set? Visualize by drawing the Venn diagram that represents this example.

***Note:** We did this example as a sample of 100 in Chapter 7.

Example: A consumer service research group studied 50 new car dealers in a city. Of the 50 surveyed 26 had good service records and of these 16 had been in service for ≥ 10 years. Of all the dealers, 30 had been in service less than 10 years. Create a contingency table to describe this example.

	Good Service (G)	**Bad Service (G′)**	
≥ 10 years	16		
< 10 years [(≥10)′]			30
	26		50

a) Redo this contingency table/two-way table to represent probabilities, instead of frequency counts (Ch. 7).

b) What's the probability of randomly selecting a dearlership with good service?

c) Of selecting a dealership with bad service or in business over 10years?

***Note:** You can use the chart to see the overlap and you can literally use the $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ or you can add up all the inner boxes of the pertinent rows and columns.

d) What is the probability of the complement of, good service or less than 10 years of service?

***Note:** Here's where the knowledge comes into play! See the note on the Venn Diagram example on the last page!

Example: Make a contingency table and change it to probabilities based on the following information:

A study of 350 consumers' smoking habits includes 200 married people (54 of whom smoke), 100 divorced people (38 of whom smoke) and 50 people who have never been married (11 whom smoke).

Example: Use the above table to find:

a) The probability that a randomly chosen person is divorced or a smoker

b) The probability that a randomly chosen person is single or does not smoke

When there is no overlap between events we can use a frequency table to find probabilities since relative frequencies are probabilities. The following is a frequency table representing the number of occurrences of sums of numbers in the roll of 2 dice.

Example 5: Given the frequency table for the telephone call data

Sum of 2 Dice	Frequency
2	1
3	2
4	3
5	4
6	5
7	6
8	5
9	4
10	3
11	2
12	1
Total	36

a) What is the probability that a random roll will result a sum of 4?

b) The probability that the roll will result in a sum less than 4?

c) The probability that the roll will result in a sum of at least 3?

d) The probability of a sum less than 5 or 10 or greater?

Note: The key here is to remember that relative frequencies are probabilities!! Frequency table classes are mutually exclusive, so $P(A \cup B) = P(A) + P(B)$
***Note**: These counts, frequencies, can be obtained by listing all the possible simple events in the roll of 2 dice. After listing the simple events, redefine the events as sums and counting the possibilities for each. These can also be counted using multinomial probabilities (see Ch. 11).

Our last discussion will be on a concept that many us will probably find interesting if we enjoy gambling! It's the concept of **odd against** an event occurring.

Odds Against an Event $\dfrac{P(A')}{P(A)}$ or $P(A'):P(A)$

Your book also discusses the **odds for** an event occurring.

Odds For an Event $\dfrac{P(A)}{P(A')}$ or $P(A):P(A')$

Some facts about **odds against** an event occurring:

1) Add up the number in the numerator & denominator [first & second is written as P(A′):P(A)] of an odds ratio and you will get the total number of trials.
2) The top(first) number is the number of ways the event won't occur. The number of ways A′ occurs.
3) The bottom (second) number is the number of ways the event will occur. The number of ways A occurs.

Some facts about **odds for** an event occurring:

1) Add up the number in the numerator & denominator [first & second is written as P(A):P(A′)] of an odds ratio and you will get the total number of trials.
2) The top(first) number is the number of ways the event will occur. The number of ways A occurs.
3) The bottom (second) number is the number of ways the event will occur. The number of ways A′ occurs.

Example: The odds against selecting a left-handed person are 9:1, so this means:

In 10 chances
9 aren't left-handed (they're right-handed)
1 is left-handed

Looking at this notationally:
A = A left-handed person
A′ = A right-handed person (not left-handed)
$P(A) = {}^{1}/_{10} = 0.1$
$P(A') = {}^{9}/_{10} = 0.9$

Your Turn: What are the odds against choosing a man in a room where there are 12 males and 18 females. Start by defining the event A, A′, P(A), P(A′) and then finding the odds. Experiment with just looking at the # of ways A′ can occur to the # of ways A can occur as well as P(A′):P(A).

Let's look at an example from biology and a way of finding classic probability based upon Mendel's Punnett Square.

Example: What are the odds against having a blue-eyed child if both parents have brown eyes based upon the gene combination of brown/blue?

<u>§8.3 Conditional Probability, Intersection & Independence</u>
<u>Multiplication Rule: Basics</u>

The first thing to be discussed in this section is the idea of independence of events.

> *Events are said to be <u>independent</u> if the occurrence of one doesn't rely upon the occurrence of the other.*

Example 1: Let's say that we roll a single die 2 times. Let event A be getting 1,2,3,4 on the first roll and event B getting a 4,5,6 of the second roll.

Solution:

Since what happens during the first roll will not effect what happens on the second roll, events A & B are independent events.

Example 2: Two cards are to be drawn from a deck of 52. If we replace the card after each draw and we consider event A = {king on first draw} and B = {king on second draw}

Solution:

Due to the replacement of the card drawn in the first draw before the second draw, the events A & B are independent.

Note: This is called sampling with replacement which always results in independent events.

Example 3: Two cards are to be drawn from a deck of 52. If we do not replace the card after each draw and we consider event A = {king on first draw} and B = {king on second draw}

Solution:

Since the card drawn in the first draw is no longer available, the probability of drawing that card changes and thus the second event is dependent upon the first. This is an example of dependent events.

Note: This is known as sampling without replacement, which results in dependent events.

This leads us to the idea of probabilities of events that occur in succession:

$$P(A \text{ and } B) = \begin{cases} P(A) \bullet P(B) & \text{if the events are independent} \\ P(A)P(B|A) & \text{if the events are dependent} \end{cases}$$

Example 4: What is the probability of getting a 1,2,3,4 on the first roll of a die and then a 4,5,6 on the second roll?

Solution:

In order to solve this problem we must first find the probability of each of the simple events A = {1,2,3,4} and then the probability of the simple event B = {4,5,6}. These are the probabilities of getting a 1 or a 2 or a 3 or a 4 (add the individual probabilities found from classic probability). After finding these probabilities then we multiply them. This is the same problem that we encountered in Chapter 7, but instead of counts we are dealing in probability.

Example 5: What is the probability of drawing a king from a deck of cards on the first draw and then a king on the second draw if the first king is not replaced.

Solution:

In order to solve this problem we must rely on classic probability for the first draw and then again rely upon classic probability considering that there is one less card in the deck and one less king among those cards.

Note: When finding these probabilities we are assuming that the event did happen! So the probability of getting a king is $^4/_{52}$, because there are four chances in 52.

Example 6: What is the probability that when drawing a marble from an urn with 3 red marbles and 5 black marbles that we get a red marble on the first draw and a black marble on the second draw assuming:

a) that the 1st marble is replaced before 2nd marble is drawn

b) that the 1st marble is not replaced before the 2nd marble is drawn

Note: Remember we are assuming that we are getting what we want – The probability that the first is red is the probability that a red is drawn and there are 3 chances in 8 marbles that it is red.
Note2: Notice that the only thing that changes in P(B|A) in part b) is that we have one less marble to choose from, but we still have 5 black marbles, unlike ex. 6 where not only the number of cards changes but also the number of kings available to be drawn.

Example: What is the probability that when drawing a marble with replacement from an urn with 3 red marbles and 5 black marbles that we get a black marble on three consecutive draws.

Note: Because of the replacement the probability remains the same and is multiplied by itself the number of times the event is expect to occur. As we know from algebra repeated multiplication can be shown with exponents, so another way of indicating the probability of a repeated event is $P(A)^n$ where n is the number of times the event is repeated.

Example: What is the probability of choosing 3 people with the same birthday?

Note: The probability here is a repeated event, but it is not $P(A)^3$, as you might expect. First we must nail down the birthday, so the first person "doesn't count" (actually the probability is 1 since their birthday can be any of the 365 days in a year), and then we want the probability that the remaining 2 have the same birthday, which is where the repeated probability comes in. This is $P_{n,r}$ divided by n^r

Example: What is the probability that 3 people are born on Thursday?

Note: This is not the same as 3 people having the same birthday, since we have the same day being a given already. This is $P(A)^3$.

Example: Four students give the excuse that they had a flat tire on the day of the exam. Find the probability that they all had the same tire go flat.

Note: Remember the birthday problem of being born on the same day. Well, this is the same problem. The first has to choose and then all others have to follow suit. So $P_{4,1}$ ways to choose out of 4^4 choices.

Example: If 5 people are randomly chosen, what is the probability that at least 2 have the same birth month?

Note: This is really the same as the complement none having the same birth month. PERMUTATION PROBLEMS SINCE ORDER MATTERS.

Let's investigate conditional probability further. Remember conditional probability operates under the assumption that something has already occurred – the given condition. Be on the look out for it because it may not always jump out and grab you.

Conditional Probability Rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Let's look to our contingency tables or two-way tables:

Example: Married/Divorced Example

	Married	Divorced	Single	
Smoker	54	38	11	103
Non-Smoker	146	62	39	247
	200	100	50	350

a) Create a probability table from this table.

b) What is the probability that a randomly chosen person smokes and is divorced?

c) What is the probability that a randomly chosen smoker is divorced?

d) What is the probability that a randomly chosen divorcee smokes?

e) What is the probability that a two randomly chosen people are both married and smoke?

***Note:** These probabilities can be found in two ways using a contingency table. One way is to use the rule at face value. The second is to take the cell of intersection and divide it by the column or row table representing the given condition. This 2nd way of looking at it is still just the definition if you refer back to the meaning of each of the cells using probability notation.

Example: What is the probability that a randomly chosen person is a smoker given they are a non-smoker?

Note: Since being a smoker and a non-smoker are mutually exclusive events, there is zero probability of choosing a smoker and a non-smoker so this probability is zero.

Example: Car Dealer Example

	Good	**Bad**	
≥ 10	16	4	20
< 10	10	20	30
	26	24	50

a) Find the probability that a randomly chosen business is good or has been in business less than 10 years.

b) Find the probability that a randomly chosen business that has been in years 10 or more years, is bad?

c) Find the probability that a two randomly chosen business' both have good service.

d) Find the probability that a randomly chosen business has good service given it has good service.

Note: This is a silly question. You are 100% certain you'll get good service if you are only choosing from those with good service!

The next problems have the repeated multiplication property applied in a slightly variant manner. They rely on having a choice of any of those possible in the first instance and then reduction of choices in the following events.

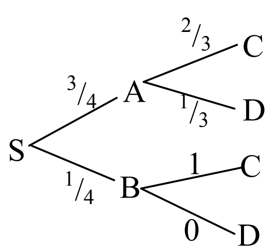
Testing for Independence
Events are independent if the probability of their intersection is the same as the product of their probabilities. This follows from the multiplication rule.

Example: Use the table above to test to see if good service and being in service 10 or more years is independent.
Solution: *To do this we need the probability of the intersection and the probability of each event. If $P(G \cap \geq 10) = P(G) \cdot (P \geq 10)$ then the events are independent and if they don't equal then they are dependent.*

Example: You try on the smoker data. Are smoking and being divorced independent?

Probability Trees
We drew tree diagrams in chapter 7 to help us count and then we used those same tree diagrams in the first section of chapter 8 to help us to find probabilities, but they can be used to find probabilities as well. Here is an example that will show us the basics of probability trees.

Example: The following tree shows the probabilities for events A, B, C & D



a) Find $P(A \cap C)$

Note: To find this we multiply the "limb" probabilities.

b) Find $P(C)$

Note: This is the $P(A \cap C)$ or $P(B \cap C)$

Example: A store receives a shipment of 4 radios and one of them is defective. Find the probability that a customer buying 2 radios will not get a defective one. Draw a tree diagram and use the ideas of conditional probability to fill in the branches.

Note: You should have a tree that looks just like the one in the last example!

Example: Let's create a tree for problem #61 p. 410 of the text and use it to answer part A).

A salesperson will receive a bonus based upon the following game.
1 - \$1 bill, 2 - \$5 bills and 1 - \$20 bill are placed in a box and the salesperson grabs a bill from the box without replacement until they grab a \$20 at which point the game ends. The sum of their bills multiplied by \$1000 is then their bonus. What is the probability that the salesperson will get a bonus of \$26,000?

§8.4 Bayes' Formula

After the discussions about probability trees and calculating conditional probabilities in the last section, this rule will make it easy to find the probability of an outcome regardless of the occurrence on the first event, which many times will be tricky to find since we won't have all of the pertinent information.

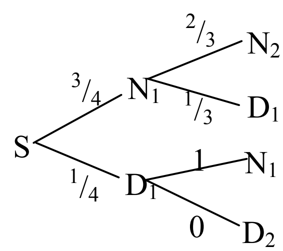
First, a note about terminology. Let's return to the radio example from above. The end of the first branch symbolically gives:

$P(1^{st}\text{ Not Defective } \cap\ 2^{nd}\text{ Not Defective})$

$= P(2^{nd}\text{ Not Defective} \mid 1^{st}\text{ Not Defective})\bullet P(1^{st}\text{ Not Defective})$

$= {}^{2}/_{3} \bullet {}^{3}/_{4} = {}^{1}/_{2}$

Just as we would have calculated in the last section



If we were interested in the probability that the 1st one chosen was not defective (and we don't know whether the 1st was or was not defective) based on the fact that the 2nd was not defective, that would be based on:

$$P(2^{nd}\text{ Not Defective}) = P(N_1 \cap N_2) + P(D_1 \cap N_1)$$

Now, by conditional probability rule we see that

$$P(1st\text{ Not Defective} \mid 2^{nd}\text{ Not Defective}) = \frac{P(N_1 \cap N_2)}{P(2^{nd}\text{ Not})}$$

and we saw that $P(2^{nd}\text{ Not Defective}) = P(N_1 \cap N_2) + P(D_1 \cap N_1)$

therefore,

$$P(1st\text{ Not Defective} \mid 2^{nd}\text{ Not Defective}) = \frac{P(N_1 \cap N_2)}{P(N_1 \cap N_2) + P(D_1 \cap N_1)}$$

and since we don't know P(Event A ∩ Event B) directly and we know the following,

$P(N_1 \cap N_2) = P(N_2 \mid N_1)P(N_1)$ since $P(N_2 \mid N_1) = \dfrac{P(N_1 \cap N_2)}{P(N_1)}$

$P(D_1 \cap N_1) = P(N_1 \mid D_1)P(D_1)$ since $P(N_1 \mid D_1) = \dfrac{P(D_1 \cap N_1)}{P(D_1)}$

this leads to the restatement

$$P(\text{1st Not Defective} \mid 2^{nd} \text{ Not Defective}) = \frac{P(N_2 \mid N_1)P(N_1)}{P(N_2 \mid N_1)P(N_1) + P(N_1 \mid D_1)P(D_1)}$$

hence,

$$P(\text{1st Not Defective} \mid 2^{nd} \text{ Not Defective}) = \frac{{}^3/_4 \bullet {}^2/_3}{{}^3/_4 \bullet {}^2/_3 + {}^1/_4 \bullet 1} = \frac{{}^1/_2}{{}^1/_2 + {}^1/_4}$$

$$= {}^1/_2 \bullet {}^4/_3 = {}^2/_3$$

Note: You can actually arrive at this same probability through sample space! There are the following simple events: ND, NN, DN and 2 of those 3 end in a not defective.

What we have just shown here is a concept referred to as **Bayes' Formula**.

Bayes' Formula

If $U_1, \ldots, U_n$ are n mutually exclusive events from S, a sample space and E is an event from that sample space and $P(E) \neq 0$, then,

$$P(U_1 \mid E) = \frac{P(U_1 \cap E)}{P(E)}$$

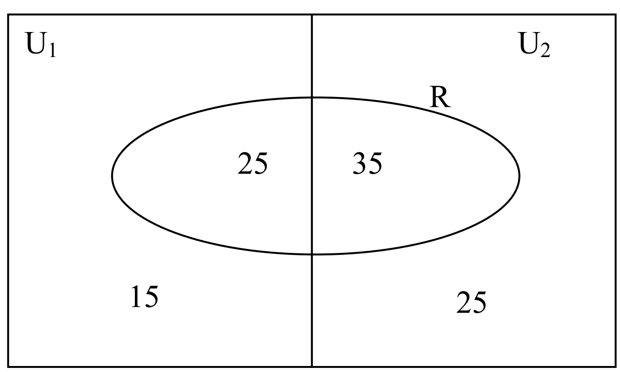
$$= \frac{P(U_1 \cap E)}{P(U_1 \cap E) + P(U_2 \cap E) + \ldots + P(U_n \cap E)}$$

$$= \frac{P(E|U_1)P(U_1)}{P(E|U_1)P(U_1) + P(E|U_2)P(U_2) + \ldots + P(E|U_n)P(U_n)}$$

Similarly for events U_2, U_3, etc.

Not only can we use tree diagrams to find probabilities using Bayes', but we can use a Venn Diagram as well. Here is an example from our book.

Example:



a) $P(R)$ b) $P(U_1 \cap R)$ c) $P(U_2 \cap R)$ d) $P(U_1 \mid R)$

Note: There are 2 ways to compute d).

Now let's apply our knowledge. First we will build a tree diagram as in the last section and then we will use it to compute our probabilities.

Example: Let's do #48 on p. 417

A company has rated 75% of its employees as satisfactory and 25% as unsatisfactory. Records indicate that of those that were satisfactory, 80% had previous work experience while of those that were not satisfactory, only 40% had previous work experience.

a) Create a probability tree with S & S′ and W & W′.

b) If a person with previous work experience is hired, find the probability that the person is a satisfactory employee.

Note: The branches that lead to the condition of having work experience through being a satisfactory employee divided by the sum of all branches that lead to having work experience. We are finding the P(S|W)

§8.5 Random Variables, Probability Distribution and Expected Value

A random variable (r.v.) can be either continuous or discrete. It takes on the possible values of an experiment. It is usually denoted: x when discussing values; X when describing the outcomes

Example: a) What are the values x can take on for the roll of a single die? Is this a discrete or continuous r.v.?

b) What are the values x can take on for the altitude of an airplane that takes off from San Francisco airport, assuming that it does not crash? Is this a discrete or continuous r.v.?

c) Suppose a coin is tossed twice so that the sample space, S, is the following set {HH, HT, TT, TH}. Let X represent the number of heads that come up. What are the possible values of the r.v. x?

A probability distribution is a function that describes the probability associated with each value of a random variable.

Example: Describe the probability distribution associated with the rolls of a single die. In order to find the distribution you must first find the possible values of the r.v. x and then based upon classic probability you will find the probability of obtaining those values.

****Note:** There is a functional relationship between the values of a r.v. and the probabilities associated with them.

Example: Describe the probability distribution associated with the number of heads obtained when a coin is tossed twice.

We should now discuss the fact that associated with every probability distribution there are certain rules which must be adhered to. They are as follows:

1. $\sum P(x) = 1$

2. $0 \le P(x) \le 1$

Example: a) Notice in the rolled die experiment where X = # on the die that the $\sum P(x) = 1$ and that $0 \le P(x) \le 1$

b) Show the same holds true for the coin example

Example: Determine if the following is a probability distribution and indicate which of the rules have been violated if it is not a pdf.

a)

x	1	2	3
f(*x*)	¼	¼	¾

b)

x	0	1	2
f(*x*)	- ¼	1	0

A probability histogram is a special case of a relative frequency histogram. It shows each probability as a rectangle whose area is equivalent to the probability of the random variable's value. As such the area under the "curve" is always equal to one. The following is a summary of the characteristics of a probability histogram.

1. Every bar is centered over a value of a random variable.
2. Every bar is one unit wide.
3. Every bar touches the one next to it.
4. The height of a bar is equivalent to the probability of the r.v.'s value
5. The area in each bar is equivalent to the probability (2 & 4 multiplied)
6. The sum of the areas under each bar is always 1. Said another way the area under the "curve" is always one.

Example: a) Draw a probability histogram for the die rolling example.

b) Draw a probability histogram for the coin toss example.

The **expected value** of a random variable is another name for the mean of a distribution. It can be thought of as the average value achieved in the long run of an experiment.

Finding the Expected Value of a PDF

$$E(X) = \sum x \cdot P(x)$$

We can then extend the expected value to games of chance and other places where money could be involved.

Example: A game is to be played with a single dice that is assumed to be fair. In the game, a player wins \$20 if a 2 turns up, \$40 if a 4 turns up, loses \$30 if a 6 turns up and neither wins nor loses (\$0) if any other number turns up. Follow the step by step process of finding the expected profit in playing such a game, by answering each of the following questions.

a) Give the PDF for rolling a single dice.

b) Use the PDF in a) to create the PDF for our game.
Hint: X =Winnings and the probabilities derive from a)

c) What is the expected profit, in the long run, from playing this game?

Your Turn: In a lottery there are 200 prizes worth \$5, 20 prizes worth \$25, and 5 prizes worth \$100. If there are 10,000 tickets sold, what is the expected winnings for this lottery? BTW, the expected winnings would be considered the expected price to pay for the ticket!