§7.2 Sets

A set (in probability this is an **event**) is a collection of things (objects). We use a capital letter to denote a set.

An element is a member (in probability this is a **simple event**) of a set – something that belongs to the collection of things. We often use a lower case letter that corresponds to the capital letter denoting the set to represent the elements of a set.

We describe sets by listing their elements in braces (roster form) or by giving their properties in braces (set builder notation). Remember that a variable represents an unknown and is typically the same lower case letter used as the capital used to denote the set itself. A third way to denote a set is with a rule – basically a function.

Example 1: The vowels of the English alphabet

$A = \{a,e,i,o,u\}$ Roster Form

$A = \{x \mid x \text{ is a vowel}\}$ Set Builder Notation

$\uparrow$ Read as "such that"

Example 2: The outcomes of a roll of a single die

$B = \{1,2,3,4,5,6\}$

$B = \{x \mid x \in W,\ 1 \le x \le 6\}$

$\uparrow$ Read as "is an element of"

Example 3: The outcomes of a roll of a pair of dice

$C = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6), (2,1),(2,2),(2,3),(2,4),(2,5),(2,6),$
$(3,1),(3,2),(3,3),(3,4),(3,5),(3,6), (4,1),(4,2),(4,3),(4,4),(4,5),(4,6),$
$(5,1),(5,2),(5,3),(5,4),(5,5),(5,6), (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$

Note: *(1,6) & (6, 1) are both listed b/c they are unique rolls if you consider each die to be different (which you should).*

$C = \{(x, y) \mid x,y \in 1,2,3,4,5,6\}$

Example 4: The set D, of all numbers such that $x^2 + 3x = -2$

When a set A belongs to a set B because all A's elements are members of set B it is called a subset and written $A \subset B$ (also read as A is contained in B)

Example 5: If $A = \{x \mid x \text{ is a vowel}\}$ and $B = \{x \mid x \in A,B,C,\ldots\}$

$\uparrow$ Read as "and so on"

In this case A is contained within in B, therefore $A \subset B$

Another way of looking at this is that A is a subset of B. When some or all of the elements of one set are contained in the other set the smaller set is said to be a subset of the larger (equal sets can also be called subsets, but have the additional property that all elements in one set are also in the other, making them equal).

Example 6: If $A = \{x \mid x \text{ is a vowel}\}$ and $C = \{a,e,i,o,u\}$

In this case $A \subset C$ and $C \subset A$ and as previously noted when this is the case the two sets are equal. $A = C$

In set theory there is what is called the <u>universe</u> (in probability theory this is called the <u>**sample space**</u>) which is some particular set of things (note in Statistics this is our population). We denote a universe with a script U (in Statistics the universe is also called our sample space and a script S is used to denote it).

A set that contains no elements is called a <u>null set</u> or an <u>empty set</u> and is written as $\varnothing$, ϕ or { }. If you use braces to indicate an empty set, make sure that you do not put either of the other symbols for an empty set inside the braces because it is no longer an empty set when this is done! You need to know that the null set is a subset of every set.

Example 7: If $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, then $\{\ \} \subset A$ is a true statement.

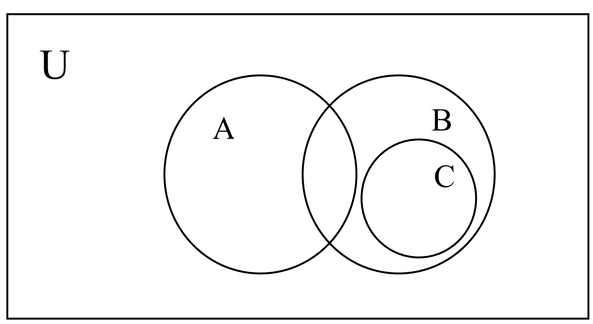
Example 8: If we consider U to be the $\Re$ (all real numbers) then

$$G = \{x \mid x^2 = -1\} \text{ is a null set}$$

Note: *A rule is used within set builder notation in this example.*

Example 9: Let the universe be all months with 31 days, then G = {February} is {} since February doesn't contain 31 days and is therefore not in the universe.

We can show sets with a <u>Venn Diagram</u> which is a visual/graphical representation of showing the abstract concept of sets.

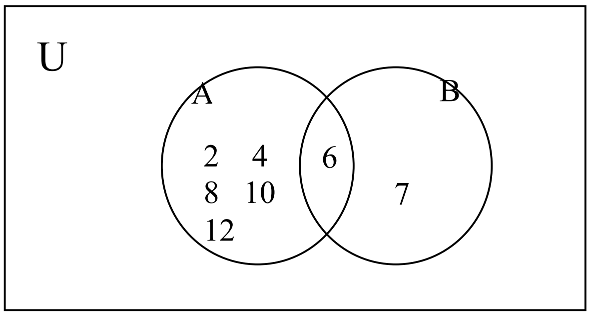


The universe is shown as a rectangle with a U in the upper left corner. Sets that belong to the universe are shown as circles with their defining capital letter inside. Sets that overlap, sets that both contain certain elements, are shown as overlapping circles (as in A & B) and sets that do not contain any like members are shown as non-overlapping (as in A & C), and if a set is a subset of another it is entirely contained within the larger set (as in B & C).

This leads to some vocabulary that is important in set theory and how it is shown using Venn Diagrams.

The **union** of sets A & B means everything that belongs to A or B or both. It is denoted as $A \cup B$.

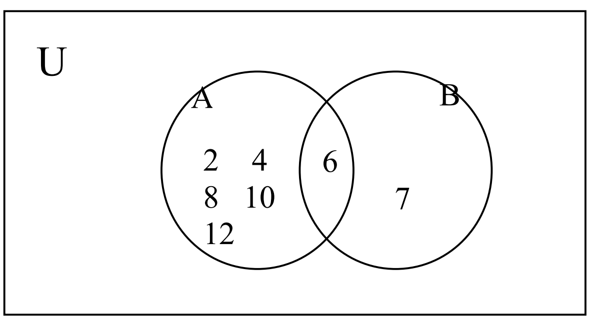
Example 10: U = Outcomes of a pair of dice
$A = \{a \mid \text{sum is even}\}$ & $B = \{b \mid \text{sum is 6 or 7}\}$



$A \cup B = \{2, 4, 6, 7, 8, 10, 12\}$

The **intersection** of A & B means everything that belongs to both A **and** B. It is denoted as $A \cap B$.

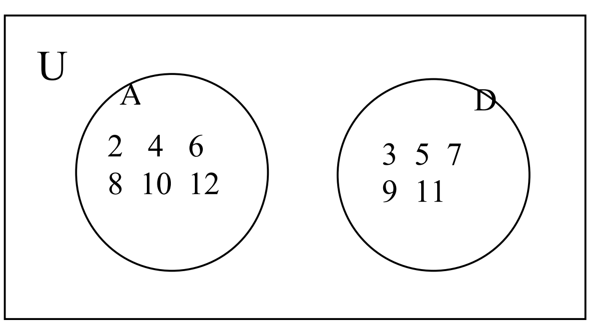
Example 11: Using the same sets in example 10



$A \cap B = \{6\}$

If $A \cap B = \varnothing$, then two sets are called disjoint or mutually exclusive. (In probability an **event** and its **complement** are mutually exclusive.)

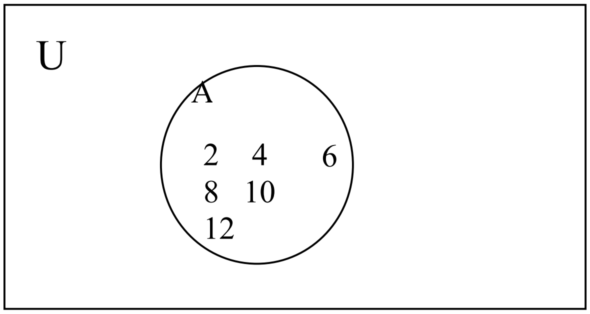
Example 12: U = Outcomes of a pair of dice
$A = \{a \mid \text{sum is even}\}$ & $D = \{d \mid \text{sum is odd}\}$



$A \cap D = \varnothing$

The **complement** of A is all elements which belong to U but not to A. Denoted as A′.

Example 13: U = Outcomes of a pair of dice
A = {a | sum is even}
What is the complement of A (use symbols to denote)?



Example 14: U = {men & women}
A = {men}
Find the complement of A (use symbols to denote and give as a set).

Now, for an example that applies Venn Diagrams.

Example 15: In a sample of 100 a randomly chosen families 86 owned an LCD TV, 35 owned a Plasma and 29 owned both types. How many families own either an LCD or a Plasma? Visualize by drawing the Venn diagram that represents this example.

§7.3 Basic Counting Principles

If A and B are disjoint sets then the number of elements in the sets in the number of elements in A, denoted n(A) and the number of elements in B, denoted n(B).

Addition Principle for Disjoint Events

$$n(A \cup B) = n(A) + n(B)$$

If the events have overlap then the addition principle for counting must account for overlap.

Addition Principle (for counting)

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Recall from set theory that this means that A or B or Both occur.

There are two ways of visualizing the probability of a compound event:

1) Venn Diagrams
2) Tables (Two-Way or Contingency)

Our last example in §7.2 displayed the method for overlapping sets. Here is another example.

Example: A survey of 800 small businesses indicates that 250 own a video conferencing system, 420 own projection equipment and 180 own video conferencing and projection equipment. Draw a Venn diagram to show this and then create a 2-way table that shows video conferencing (C) & non-video conferencing (C′) in the columns and projection (P) & non-projection (P′) in the rows. (See Table 1 on p. 350 of your book.)

<u>Multiplication/Counting Rule</u>: # of Outcomes of ≥ 2 events occurring.

Event 1: "m" ways of occurring
Event 2: "n" ways of occurring
ETC.

# of outcomes = m • n

Example: Find the number of possible outcomes

a) If I toss three coins, how many possible outcomes are there? Draw a tree to verify that your answer is correct.

b) How many 4-digit codes are possible from the digits 0-9?

c) From the 4 weeks in a month that I teach my Algebra classes I wish to choose the possible outcomes for my quizzes. Each week I teach Algebra, Monday-Thursday. In four weeks how many possible outcomes are there?

d) From the 4 weeks in a month that I teach my Algebra classes I wish to choose the possible outcomes for my quizzes. Each week I teach Algebra, Monday-Thursday. In four weeks how many possible outcomes are there if I don't wish to have a quiz on the same day of the week?

Your Turn: a) Event A is rolling a 1, 2, 3, 4 on the first roll of a dice. Event B is rolling a 5 or 6 on the second roll of the dice. How many possible outcomes are there for A and B to occur?

b) Repeat b) above, but with the restriction that no digit can be repeated.

§7.4 Permutations & Combinations

Factorial Rule: *Arrangement/Sequences* of "n" things in order

$$\mathbf{n! = n(n-1)(n-2)(n-3)(n-4)\ldots[n-(n-1)]}$$

Example: Let's logic this example out to see where this Factorial Rule comes from .

How many different arrangements of the digits 0-9 are there?

1) How many choices do you have for the first digit?

2) Now the digit that you first used is no longer a choice, so how many choices are left for the second one?

3) And the pattern will continue… Our answer then relies on the **Counting Rule**. The answer from 1) is the ways the first event can occur, 2) the number of ways that the second event can occur, etc. This is where the **Factorial Rule** comes from.

Example: There are 4 wires (red, green, blue and yellow) that we wish to attach to a circuit board and we want to know how many arrangements of the wires there could be.

Note: *The wires are unique and the order in which they are attached is significant, and that is what makes this different than the mere Multiplication Principle.*

Permutations Ways to arrange/sequence/permute "n" different items "r" at a time

$${}_nP_r = P_{n,r} = \frac{n!}{(n-r)!}$$

n = # of unique items
r = # chosen at a time

Note: *Order matters!! Called Permutations, Arrangements & Sequences!*
Note2: *If we permute all items, n items,* $P_{n,n} = n!$

Example: If 3 people are randomly chosen, how many arrangements are there such that none were born on the same day of the week?

Note: *This is the Permutation! Each person must be born on a unique day. Logic can get you through this problem too. The number of ways the first person could have their birthday is 7, then the next would have 6 to choose from and finally the 3^{rd} would have 5 days to choose from. The product of these would be the # of outcomes.*

Example: A classic excuse for a student to miss an exam is to have had a flat tire on the way to take the exam. If 4 student's claim they had a flat, how many possible ways are there for every student to claim they had a flat on the left-front tire? Draw a tree diagram for this example too.

Your Turn: There are currently 10 people in the math club. If 3 are to be randomly selected to fill the positions of treasurer, secretary, and president, how many possible outcomes can there be?

Note: *Order is vital here because Sue being secretary is different than Sue being president or Sue being treasurer. This a key distinction to make when it comes to distinguishing between permutations and the next concept combinations.*

<u>Combinations</u> Ways to combine "r" items from "n" different items

$$_nC_r = \frac{n!}{r!\,(n-r)!}$$

n = # of different items
r = # of items chosen at a time

Note: *Order doesn't matter! Same items arranged are the same. CAT means same as CTA means same as ACT means the same as TAC means the same as ATC means the same as TCA.*

Example: This example is to illustrate the difference between a permutation and a combination.

There are 5 people up for nomination for the governing board of a club. The people are Joe, Alice, Sue, Elly and Bill.

a) How many ways can a governing board of 3 be chosen from these 5 people up for nomination?

Note: *This is a combination because order doesn't matter. Whether we choose Bill Sue Joe, or Sue Bill Joe or Joe Sue Bill, they are the same 3 being chosen.*

b) How many ways can the President, Secretary and Treasurer be chosen from the 5 people up for nomination?

Note: *Now order matters! Joe could be Pres., Sue could be Secretary, Bill could be Treasurer, or a different permutation -- Sue could be Pres., Bill could be Secretary, Joe could be Treasurer, which is certainly different from the first arrangement.*

Example:

I give homework assignments to my Algebra classes where I randomly choose 5 problems to grade.

a) If there are 18 problems, how many ways are there to choose 5 from among the 18?

b) If a student did 12 of the 18 problems, how many different groups of 5 did this student do?

Note: *This is asking, how many groups of 5 are in 12!*

Now we need to put together some of the counting techniques that we have learned. Remember that hearing or seeing the word and is an indicator that the multiplication principle is being used. And sometimes the best way to arrive at a problem that is asking for at least a certain number of occurrences is to add all the possible disjoint outcomes. See p. 364 of your text for more examples.

Example: A company has 7 senior and 5 junior officers. It wants to form a committee consisting of

a) four members with 1 senior officer and 3 junior officers. How many such committees can be formed?

b) four members with 4 junior officers. How many such committees can be formed?

c) four members with at least 2 junior officers. How many such committees can be formed?