Chapter 4
Introduction to Chapter 4

Set Theory

A set (in probability this is an **event**) is a collection of things.
An element is a member (in probability this is a **simple event**) of a set – something that belongs to the collection of things.

We describe sets by listing their elements in braces (roster form) or by giving their properties in braces (set builder notation). Remember that a variable represents an unknown!!

Example 1: The vowels of the English alphabet

{a,e,i,o,u} Roster Form
{x | x is a vowel} Set Builder Notation
↑— Read as "such that"

Example 2: The outcomes of a roll of a single die

{1,2,3,4,5,6}
$\{x \mid x \in W,\ 1 \le x \le 6\}$
↑— Read as "is an elememt of"

Example 3: The outcomes of a roll of a pair of dice

{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6), (2,1),(2,2),(2,3),(2,4),(2,5),(2,6), (3,1),(3,2),(3,3),(3,4),(3,5),(3,6), (4,1),(4,2),(4,3),(4,4),(4,5),(4,6), (5,1),(5,2),(5,3),(5,4),(5,5),(5,6), (6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

Note: *(1,6) & (6, 1) are both listed b/c they are unique rolls if you consider each die to be different (which you should).*

$\{(x, y) \mid x,y \in 1,2,3,4,5,6\}$

We name sets using capital letters. **Ex:** A = {x | x is a vowel}

When a set A belongs to a set B because all A's elements are members of set B it is called a subset and written $A \subset B$ (read as A is contained in B) or $B \supset A$ read as B contains A)

Example 4: If A = {x | x is a vowel} and B = $\{x \mid x \in A,B,C,\ldots\}$
↑— Read as "and so on"

In this case A is contained within in B, therefore $A \subset B$

Another way of looking at this is that A is a subset of B. When some or all of the elements of one set are contained in the other set the smaller set is said to be a subset of the larger (equal sets can also be called subsets, but have the additional property that makes them equal).

Example 5: If $A = \{x \mid x \text{ is a vowel}\}$ and $C = \{a,e,i,o,u\}$

In this case $A \subset C$ and $C \subset A$ and as previously noted when this is the case the two sets are equal.

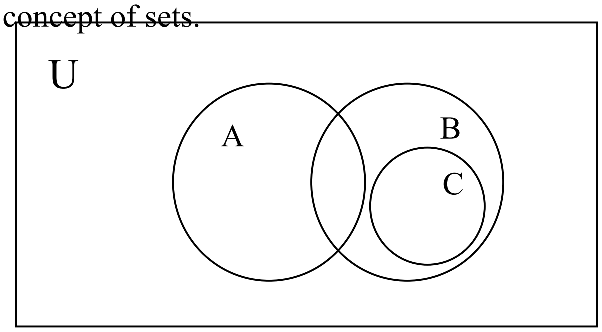
In set theory there is what is called the universe (in probability theory this is called the **sample space**) which is some particular set of things (note in Statistics this is our population). We denote a universe with a script U (in Statistics the universe is also called our sample space and a script S is used to denote it).

A set that contains no elements is called a null set or an empty set and is written as $\varnothing$, ϕ or { }. If you use braces to indicate an empty set, make sure that you do not put either of the other symbols for an empty set inside the braces because it is no longer an empty set when this is done!

Example 6: If we consider U to be the $\Re$ (all real numbers) then
$G = \{x \mid x^2 = -1\}$ is a null set

Example 7: Let the universe be all months with 31 days, then G = {February} is {} since February doesn't contain 31 days and is therefore not in the universe.

We can show sets with a Venn Diagram which is a visual/graphical representation of showing the abstract concept of sets.

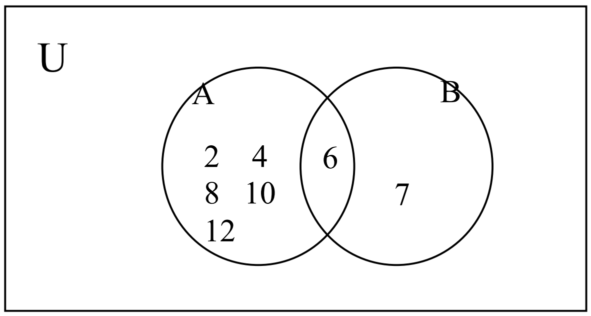


The universe is shown as a rectangle with a U in the upper left corner. Sets which belong to the universe are shown as circles with their defining capital letter inside. Sets that overlap are shown as overlapping circles (as in A & B) and sets that do not contain any like members are shown as non-overlapping (as in A & C), and if a set is a subset of another it is entirely contained within the larger set (as in B & C).

This leads to some vocabulary that is important in set theory and how it is shown using Venn Diagrams.

The <u>union</u> of sets A & B means everything that belongs to A or B or both. It is denoted as $A \cup B$.

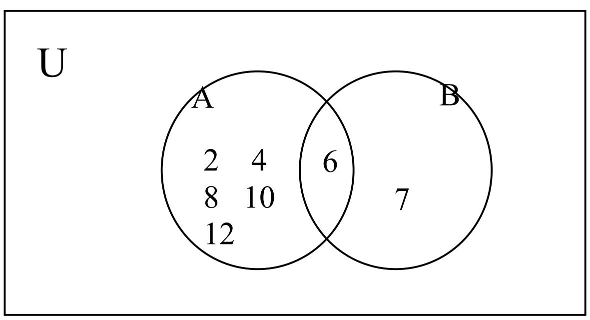
Example 8: U = Outcomes of a pair of dice
$A = \{a \mid \text{sum is even}\}$ & $B = \{b \mid \text{sum is 6 or 7}\}$



$A \cup B = \{2, 4, 6, 7, 8, 10, 12\}$

The <u>intersection</u> of A & B means everything that belongs to both A **<u>and</u>** B. It is denoted as $A \cap B$.

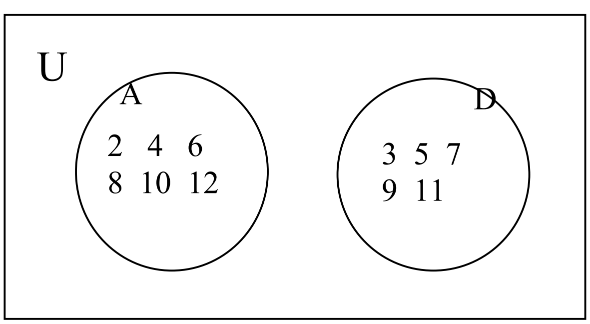
Example 9: Using the same sets in example 8



$A \cap B = \{6\}$

If $A \cap B = \varnothing$, then two sets are called <u>disjoint</u> or <u>mutually exclusive</u>. (In probability an **<u>event</u>** and its **<u>complement</u>** are mutually exclusive.)

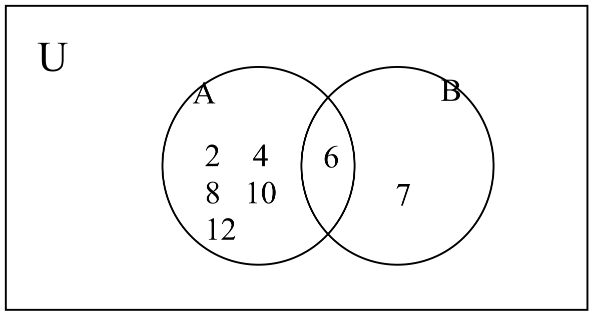
Example 10: U = Outcomes of a pair of dice
$A = \{a \mid \text{sum is even}\}$ & $D = \{d \mid \text{sum is odd}\}$



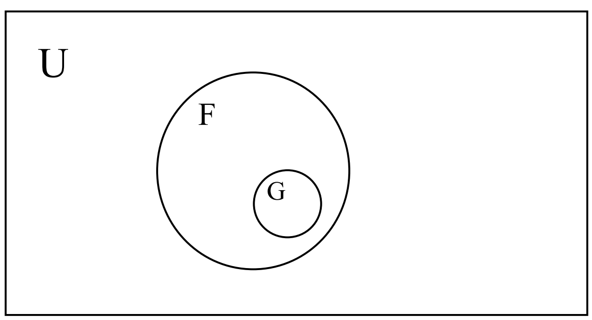
$A \cap D = \varnothing$

The difference of A & B is all elements which belong to A but not to B. Denoted as A – B.

Example: U = Outcomes of a pair of dice
A = {a | sum is even} & B = {b | sum is 6 or 7}
What is the difference between A and B (use symbols to denote)?
What is the difference between B and A (use symbols to denote)?



Example: U = Outcomes of a pair of dice
F = {x | sum is even} & G = {(x_1,x_2) | $(x_1,x_2) \in$ (1,3), (2,2), (3,1)}
What is the difference of F and G (use symbols to denote)?



In the last example F – G also has another name. The name for F – G is the **complement** of G relative to F, since $G \subset F$. Another way of thinking of the complement is everything that is not in G relative to another set (typically we will be thinking of it relative to the universe). It is denoted as: $\overline{G}$. (Note: Your book discusses a complement with a different notation; G and G^c are complements.)

Example: U = {men & women}
A = {men} & B = {women}
Find the complement of A (use symbols to denote and give as a set).
Show the sets using a Venn Diagram.

§4.1 What is Probability?

Your book begins by defining probability, but probability can't be discussed until we have thoroughly defined a sample space and an event, so I'd like to talk about those first.

The ideas of set theory come into play when we talk about probability because we must have in mind a **sample space** (a universe), define an **event** (a subset of the universe) and then be able to talk about a **simple event** (an element of the subset of the universe).

A **sample space** is all possible outcomes (simple events) for an experiment. We usually denote a sample space with a capital S or $\mathcal{S}$.

Example: All the ordered pairs that result from the roll of a pair of dice yield a sample space.

$$\mathcal{S} = \{(x,y) \mid x,y \in 1,2,3,4,5,6\}$$

Example: The possible outcomes of the flip of a coin.

$$\mathcal{S} = \{\text{heads, tails}\}$$

Your Turn: Give the sample space for the possible outcomes for the flip of 3 coins (flipping a single coin 3 times).

***Note:** A perfect method of finding all the possible combinations is to draw a tree diagram where each level represents a coin and each branch represents one of the two possible outcomes.

These sample spaces come about in accordance with an **experiment** which we already know is a process by which we gather information. Because experiments may be looking for something in particular (the population) sample spaces must be defined accordingly.

An **event** is an outcome or collection of outcomes from an experiment. It is equivalent to a set defined within the universe in set theory (in other words a subset of S).

Example A: The sum of the pair of dice is an event.

A = {2, 3, 4, 5, 6, …, 12} describes this event

Example B: The ordered pair that represents a sum of seven on a pair of dice when 2 die are rolled simultaneously.

B = {(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}

Your Turn: Describe the event of getting 2 heads when flipping 3 coins simultaneously?

A **simple event** is any element of the sample space for which there is no further break down.

Example: For each of the above examples of events which would be considered a simple event?

A)

B)

C)

Now we're able to talk about **probabilities**. We talk about the probabilitiy of an event occurring when a sample space has been thoroughly defined. A probability is the likelihood of occurrence of an event.

Probability is the likelihood of the occurrence of an event. It is a numeric measure of this likelihood that must be a number between 0 and 1. I'm going to talk about the "heart-and-soul" of probability before we continue.

Heart-and-Soul Rules of Probability

1) All probabilities must be between zero and one
$$0 \le P(A) \le 1$$

2) If the event can't possibly happen the there is zero probability.
$$P(A) = 0$$

3) If there is only one way in which an event can occur (the event is equivalent to the entire sample space) then the probability is one.
$$P(A) = 1$$

There are 3 ways of defining probability:

1) The **intuitive** method for finding the probability of an event.
Example: The probability that another planet, like the Earth, could exist in the universe in the next 4 million years, according to Prof. Watson is less than 0.0001

2) The **relative frequency** approach for finding the probability of event A.
$P(A) = f / n$ where f = frequency of occurrence **and** n = total number of outcomes
Example: After flipping a coin 15 times, the following outcomes were noted: H, H, T, T, T, T, T, H, T, T, H, T, H, T
P(H) = 5/15 = 0.33

Note: *This data was created using EXCEL. I used* =ROUND(RAND()*(1-0)+0,0) to create 15 random digits and then I coded them using the following =IF(A1=0,"H",IF(A1=1,"T"))
Note2: *Probabilities are usually rounded to 2 significant digits (meaning 2 actual digits, not place values)*

3) The **classic approach**, based upon equally likely outcomes.

$P(A) = s / n$ where s = number of ways event A can occur **and**
n = number of simple events

Example: What is the probability of drawing a king from an unmarked deck of 52 card?

First: Define the number of simple events (there are two ways to do this for this particular problem; we'll be using the most basic approach – the cards).

Second: How many ways can you get a king

Finally: $P(K) = {}^{4}/_{52} = {}^{1}/_{13}$

Note: *A probability can be expressed as a decimal or as a fraction in lowest terms or as a percentage.*

Now, let's compare two of the approaches to finding probability – the classic and relative frequency approach and use it to illustrate something called the **Law of Large Numbers**.

Example: **a)** What does classic probability tell us the probability of getting a head is when we flip a coin?

b) What did our relative frequency approach tell us?

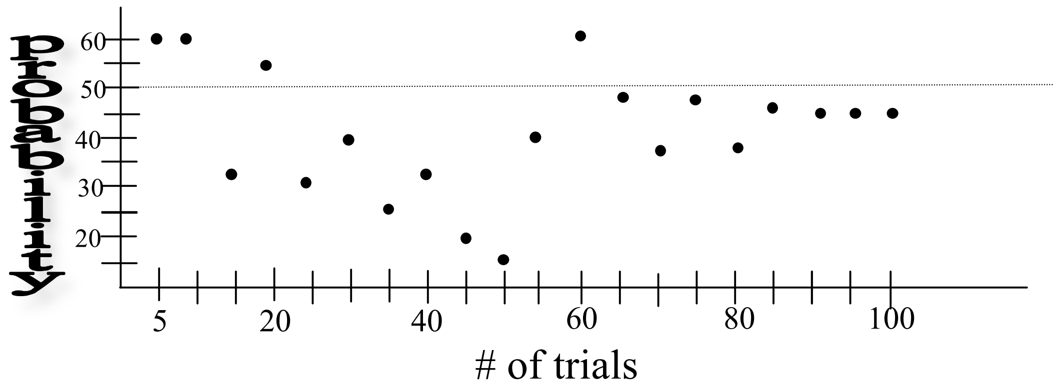
c) Why do they differ?

d) I will try to do this with the computer, in-class for you, but there is no guarantee that it will happen, so just in-case it can't here is some data and the note below indicates how I got that data using my EXCEL for Mac Office 2004:

$P(H_{20}) = 11/20 = 0.55$
$P(H_{25}) = 8/25 = 0.32$
$P(H_{30}) = 12/30 = 0.40$
$P(H_{35}) = 9/35 = 0.26$
$P(H_{40}) = 13/40 = 0.33$
$P(H_{45}) = 9/45 = 0.2$
$P(H_{50}) = 8/50 = 0.16$
$P(H_{55}) = 22/55 = 0.4$
$P(H_{60}) = 36/60 = 0.6$
$P(H_{65}) = 31/65 = 0.48$
$P(H_{70}) = 25/70 = 0.36$
$P(H_{75}) = 36/75 = 0.48$
$P(H_{80}) = 30/80 = 0.38$
$P(H_{85}) = 39/85 = 0.46$
$P(H_{90}) = 40/90 = 0.45$
$P(H_{95}) = 43/95 = 0.45$
$P(H_{100}) = 45/100 = 0.45$

Note: *I created the data with used* =ROUND(RAND()*(1-0)+0,0) to create random digits and then I coded them using the following =IF(A1=0,"T",IF(A1=1,"H")) *In addition I summed each column of 0's and 1's, thus finding the number of heads. After summing the number of heads, I used the formula =a(n+1)/n [n+1 being the row containing the sum and n being the number of trials].*

A graph of the data will nicely show what is happening:



Law of Large Numbers

In the long run, as the number of trials increase (the sample size increases) the relative frequency probability will approach that of theoretical/classic probability.

Now, we will discuss the concept of a **complement**. The complement of an event and the event itself together comprise the entire sample space. Notationally, your book differs slightly from set theory and other texts. Your book uses:

A^c to denote the complement

$\overline{A}$ typically denotes the complement

Therefore, in terms of probability:

$$P(A^c) = 1 - P(A),$$

which can really help us in defining probabilities.

Example: If a coin is flipped 1000 times and 613 heads appeared, what is the probability of getting a tail?

A = Getting a head

$\therefore$ A^c = Getting a tail since $A \cup A^c = S$

$P(A) = 613/1000 = 0.613$

$P(A^c) = 1 - P(A) = 1 - 0.613 = 0.487$

Your Turn: Ten thousand people are surveyed and 2500 are found to like the product in question. What is the probability that a person does not like the product? Use the complement to find this probability.

Our last discussion will be on a concept that many us will probably find interesting if we enjoy gambling! It's the concept of **<u>odd against</u>** an event occurring.

Odds Against an Event $\dfrac{P(A^c)}{P(A)}$ or $P(A^c):P(A)$

Some facts:

1) Add up the number in the numerator & denominator [first & second is written as $P(A^c):P(A)$] of an odds ratio and you will get the total number of trials.
2) The top(first) number is the number of ways the event won't occur. The number of ways A^c occurs.
3) The bottom (second) number is the number of ways the event will occur. The number of ways A occurs.

Example: The odds against selecting a left-handed person are 9:1, so this means:

In 10 chances
9 aren't left-handed (they're right-handed)
1 is left-handed

Looking at this notationally:
A = A left-handed person
A^c = A right-handed person (not left-handed)
$P(A) = 1/10 = 0.1$
$P(A^c) = 9/10 = 0.9$

Your Turn: What are the odds against choosing a man in a room where there are 12 males and 18 females. Start by defining the event A, A^c, P(A), $P(A^c)$ and then finding the odds. Experiment with just looking at the # of ways A^c can occur to the # of ways A can occur as well as $P(A^c):P(A)$.

Let's look at an example from biology and a way of finding classic probability based upon Mendell's Square.

Example: What are the odds against having a blue-eyed child if both parents have brown eyes based upon the gene combination of brown/blue?

§4.3 Addition Rule

This section discusses the compound event in which two or more simple events occur. Along with the compound event comes the **Addition Rule** which will allow us to find the probability of a compound event.

Addition Rule

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Recall from set theory that this means that A or B or Both occur.

There are two ways of visualizing the probability of a compound event:

1) Venn Diagrams
2) Tables (Two-Way or Contingency)

Example 1: The probability that a randomly chosen family will own a color TV is 0.86, a black and white set is 0.35 and both types is 0.29. What is the probability that a randomly chosen family will own either a color or a black and white set? Visualize by drawing the Venn diagram that represents this example.

Example 2: A consumer service research group studied 50 new car dealers in a city. Of the 50 surveyed 26 had good service records and of these 16 had been in service for ≥ 10 years. Of all the dealers, 30 had been in service less than 10 years. What's the probability of randomly selecting a dearlership with good service? Of selecting a dealership with bad service or in business over 10years? Create a contingency table to describe this example.

	Good Service	**Bad Service**	
≥ 10 years	16		
< 10 years			30
	26		50

***Note:** You can use the chart to see the overlap.

Example 3: Make a contingency table for the following scenario:

A study of consumer smoking habits includes 200 married people (54 of whom smoke), 100 divorced people (38 of whom smoke) and 50 people who have never been married (11 whom smoke).

Example 4: Now let's find

a) The probability that a randomly chosen person is divorced or a smoker
b) is single or does not smoke

We can also use a frequency table to find probabilities since relative frequencies are probabilities.

Example 5: Given the frequency table for the telephone call data

Time in Minutes	Frequency
0-3	3
4-7	10
8-11	7
12-15	8
16-19	5
20-23	2
24-27	5
28-31	2
32-35	2
36-39	1
40-43	1
44-47	2
48-51	1
Total	49

a) What is the probability that a randomly chosen person would be on the phone less than 4 minutes?

b) The probablility that they are on the phone less than 12 minutes or more than 39 minutes?

c) The probability that they are on the phone no more than 39 minutes?

d) On the phone at least 20 minutes?

Note: The key here is to remember that relative frequencies are probabilities!! Frequency table classes are mutually exclusive, so $P(A \cup B) = P(A) + P(B)$

<u>§4.4 Multiplication Rule: Basics</u>

The first thing to be discussed in this section is the idea of independence of events.

Events are said to be <u>independent</u> if the occurrence of one doesn't rely upon the occurrence of the other.

Example 1: Let's say that we roll a single die 2 times. Let event A be getting 1,2,3,4 on the first roll and event B getting a 4,5,6 of the second roll.

Since what happens during the first roll will not effect what happens on the second roll, events A & B are independent events.

Example 2: Two cards are to be drawn from a deck of 52. If we replace the card after each draw and we consider event A = {king on first draw} and B = {king on second draw}

Due to the replacement of the card drawn in the first draw before the second draw, the events A & B are independent.

Note: This is called sampling with replacement which always results in independent events.

Example 3: Two cards are to be drawn from a deck of 52. If we do not replace the card after each draw and we consider event A = {king on first draw} and B = {king on second draw}

Since the card drawn in the first draw is no longer available, the probability of drawing that card changes and thus the second event is dependent upon the first. Th3s is an example of dependent events.

Note: This is known as sampling without replacement and this results in dependent events.

This leads us to the idea of probabilities of events that occur in succession:

P(A and B) =

- P(A) • P(B) if the events are independent
- P(A)P(B|A) if the events are dependent

Example 4: What is the probability of getting a 1,2,3,4 on the first roll of a die and then a 4,5,6 on the second roll?

In order to solve this problem we must first find the probability of each of the simple events A = {1,2,3,4} and then the probability of the simple event B = {4,5,6}. These are the probabilities of getting a 1 or a 2 or a 3 or a 4 (add the individual probabilities found from classic probability). After finding these probabilities then we multiply them.

Example 5: What is the probability of drawing a king from a deck of cards on the first draw and then a king on the second draw if the first king is not replaced.

In order to solve this problem we must rely on classic probability for the first draw and then again rely upon classic probability considering that there is one less card in the deck and one less king among those cards.

Note: When finding these probabilities we are assuming that the event did happen! So the probability of getting a king is 4/52, because there are four chances in 52.

Example 6: What is the probability that when drawing a marble from an urn with 3 red marbles and 5 black marbles that we get a red marble on the first draw and a b lack marble on the second draw assuming:

a) that the 1^{st} marble is replaced before 2^{nd} marble is drawn

b) that the 1^{st} marble is not replaced before the 2^{nd} marble is drawn

Note: Remember we are assuming that we are getting what we want – The probability that the first is red is the probability that a red is drawn and there are 3 chances in 8 marbles that it is red.

Note2: Notice that the only thing that changes in P(B|A) in part b) is that we have one less marble to choose from, but we still have 5 black marbles, unlike ex. 6 where not only the number of cards changes but also the number of kings available to be drawn.

Example 7: What is the probability that when drawing a marble with replacement from an urn with 3 red marbles and 5 black marbles that we get a black marble on three consecutive draws.

Note: Because of the replacement the probability remains the same and is multiplied by itself the number of times the event is expect to occur. As we know from algebra repeated multiplication can be shown with exponents, so another way of indicating the probability of a repeated event is $P(A)^n$ where n is the number of times the event is repeated.

Example 8: What is the probability of choosing 3 people with the same birthday?

Note: The probability here is a repeated event, but it is not $P(A)^3$, as you might expect. First we must nail down the birthday, so the first person "doesn't count" (actually the probability is 1 since their birthday can be any of the 365 days in a year), and then we want the probability that the remaining 2 have the same birthday, which is where the repeated probability comes in.

§4.5 Multiplication Rule: Complements & Conditional Probability

The phrase "at least" needs to be discussed as it is frequently used and must be interpreted, in order to find probabilities. When I say at least 5, I mean 5 or more, so when I talk about the probability of at least 5, I mean the probability of 5 or more occurring.

Example 1: In flipping 3 coins at the same time, what is the probability of getting a least 1 head?

In order to solve this problem we must think about the sample space of the simple event of flipping 3 coins at the same time. After that we must think of the event A = {1 or more heads occur}. The best method of calculating this probability is not multiplication or addition rule but looking at the sample space and the ways of event A occurring.

Before discussing the next example let's discuss redefining a question in terms of the complement of "at least". For instance, if I want to know the probability of at least 1 head the complement of this event is no heads, so the

$$P(\text{at least 1 head}) = 1 - P(\text{no heads})$$

Another way that this can be used is to find the probability of less than, since less than is the complement of "at least". For instance, if I say less than 5, I means everything below 5, and not including 5, this is the complement of "at least 5".

$$P(\text{less than 5}) = 1 - P(\text{at least 5})$$

Example 2: What is the probability of getting less than 2 heads when flipping 3 coins?

In order to solve this problem first redefine less than in terms of the complement of at least 2 heads, using the sample space from ex. 1, the probabilities are then easy to find. Note that the probability can just as easily be found directly but this is not always the case, so it is helpful to know this trick as a backup plan!

Example 3: A blood testing procedure is made more efficient by combining samples of blood specimens. If samples from 5 people are combined and mixed and the mixture tests negative then we know all 5 individual are negative. Find the probability of a positive result for 5 samples combined into 1 mixture assuming a probability of testing positive is 0.015.

Thinking about the question a little, we must realize that 1 or more people would need to be positive in order for the blood mixture to test positive, but it would be unknown how many it could be as few as 1 positive, but as many as 5. This would be worded as "at least 1" is positive for a positive mixture.

Step 1: Define the question in terms of at least using probability notation.

Step 2: Define the complement of at least one.

Step 3: Find the probability that none are positive. $P(A) = 0.015$ so $P(\overline{A}) = 0.985$ and this is a repeated multiplication problem since the events are independent.

Step 4: Find the desired probability by using the compliment that none are positive.

Note: This problem takes several problems and links them together, so you may want to go through it several times until you are clear on the different components that must be linked.

Example 4: A student experiences difficulties with malfunctioning alarm clocks. Instead of using 1 alarm, he decides to use 3. What is the probability that at least one works if each individual has 0.98 chance of working?

This is the same type of logic as the last problem. First put the original question in terms of the complement of the event A. Next find the probability of none of the alarms functioning and then use that to solve the problem.

Step 1: Define the question in terms of at least using probability notation.

Step 2: Define the complement of at least one.

Step 3: Find the probability that none are positive. P(A) = so $P(\overline{A})$ = and this is a repeated multiplication problem since the events are independent.

Step 4: Find the desired probability by using the compliment that none are positive.

Now let's investigate conditional probability further. Remember conditional probability operates under the assumption that something has already occurred – the given condition. Be on the look out for it because it may not always jump out and grab you.

Conditional Probability Rule

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Let's look to our contingency tables:

Example 5: Married/Divorced Example

	Married	**Divorced**	**Single**	
Smoker	54	38	11	103
Non-Smoker	146	62	39	247
	200	100	50	350

a) What is the probability that a randomly chosen person smokes and is divorced?

b) What is the probability that a randomly chosen smoker is divorced?

c) What is the probability that a randomly chosen divorcee smokes?

d) What is the probability that a two randomly chosen people are both married and smoke?

Example 6: Car Dealer Example

	Good	**Bad**	
≥ 10	16	4	20
< 10	10	20	30
	26	24	50

a) Find the probability that a randomly chosen business is good or has been in business less than 10 years.

b) Find the probability that a randomly chosen business that has been in years 10 or more years, is bad?

c) Find the probability that a two randomly chosen business' both have good service.

The next problems have the repeated multiplication property applied in a slightly variant manner. They rely on having a choice of any of those possible in the first instance and then reduction of choices in the following events.

Example 6: If 3 people are randomly chosen what is the probability that

a) All were born on Friday? (This is no different than before, the probability that each is born on Friday multiplied by one another.)

b) None were born on Friday? (This requires the complement of being born on Friday being repeatedly multiplied.)

c) None were born on the same day of the week? (Here the pattern begins. The first person can be born on any day of the week so 7/7 choices, then the second has only 6 days on which they can be born to fit the criteria, and finally the last person can't be born on the day that 1st or 2nd were born so they only have 5 choices.)

Example 7: A classic excuse for a missed test is offered by 4 students who claim that their car had a flat tire. On the makeup test, the instructor asks the students to identify the particular tire that went flat. What the probability that they didn't have flats and all randomly chose the same flat?

Again, the first person can choose any of the 4 tires so their probability is $^4/_4$ of choosing the tire, but then each successive student must choose the same tire making the probability ¼ for each of them to choose the same tire as the first. Because that's only 3 students with probability ¼ (the first had $^4/_4$) this means that the P(A) = $(¼)^3$.

§4.6 Counting

Although this section isn't technically on our agenda I want to make sure that I highlight a few of the techniques here and define permutations and combinations, which rely on these techniques.

Multiplication/Counting Rule: # of Outcomes of ≥ 2 events occurring.

Event 1: "m" ways of occurring
Event 2: "n" ways of occurring
ETC.

$$\boxed{\textbf{\# of outcomes} = \mathbf{m \cdot n}}$$

Example: Find the number of possible outcomes

a) If I toss three coins, how many possible outcomes are there? Draw a tree to verify that your answer is correct.

b) How many 4-digit codes are possible from the digits 0-9?

c) From the 4 weeks in a month that I teach my Algebra classes I wish to choose the possible outcomes for my quizzes. Each week I teach Algebra, Monday-Thursday. In four weeks how many possible outcomes are there?

Your Turn: Event A is rolling a 1, 2, 3, 4 on the first roll of a dice. Event B is rolling a 5 or 6 on the second roll of the dice. How many possible outcomes are there for A and B to occur?

<u>Factorial Rule</u>: *Arrangement/Sequences* of "n" things in order

$$\mathbf{n! = n(n-1)(n-2)(n-3)(n-4)\ldots[n-(n-1)]}$$

Example: Let's logic this example out to see where this Factorial Rule comes from .

How many different arrangements of the digits 0-9 are there?

1) How many choices do you have for the first digit?

2) Now the digit that you first used is no longer a choice, so how many choices are left for the second one?

3) And the pattern will continue… Our answer then relies on the **Counting Rule**. The answer from 1) is the ways the first event can occur, 2) the number of ways that the second event can occur, etc. This is where the **Factorial Rule** comes from.

Example: There are 4 wires (red, green, blue and yellow) that we wish to attach to a circuit board and we want to know how many arrangements of the wires there could be.

Note: *The wires are unique and the order in which they are attached is significant, and that is what makes this different than the mere Counting/Multiplication Rule.*

<u>Permutations</u> Ways to arrange/sequence/permute "n" different items "r" at a time

$${}_nP_r = \frac{n!}{(n-r)!}$$

n = # of unique items
r = # chosen at a time

Note: *Order matters!! Called Permutations, Arrangements & Sequences!*

Example: If 3 people are randomly chosen, how arrangements are there such that none were born on the same day of the week?

Note: *This is the Permutation! Each person must be born on a unique day. Logic can get you through this problem too. The number of ways the first person could have their birthday is 7, then the next would have 6 to choose from and finally the* 3^{rd} *would have 5 days to choose from. The product of these would be the # of outcomes.*

Example: There are 3 positions available in a restaurant, counter help, wait-staff and cook. If 7 fully qualified people apply for all three of the positions how many different ways can the positions be filled.

Your Turn: At a local horseshow, 27 equally qualified entrants sign up for the Working Cow Horse competition. How many possible ways can 1^{st}, 2^{nd} , 3^{rd} and 4^{th} position be awarded.

Note: *The word different indicates that we are looking at a case where order matters! Watch for this or anything that would indicate uniqueness!*

Combinations Ways to combine "r" items from "n" different items

$${}_nC_r = \frac{n!}{r!\,(n-r)!}$$

n = # of different items

r = # of items chosen at a time

Note: *Order doesn't matter! Same items arranged are the same. CAT means same as CTA means same as ACT means the same as TAC means the same as ATC means the same as TCA.*

Example: This example is to illustrate the difference between a permutation and a combination.

There are 5 people up for nomination for the governing board of a club. The people are Joe, Alice, Sue, Elly and Bill.

a) How many ways can a governing board of 3 be chosen from these 5 people up for nomination?

__Note:__ This is a combination because order doesn't matter. Whether we choose Bill Sue Joe, or Sue Bill Joe or Joe Sue Bill, they are the same 3 being chosen.

b) How many ways can the President, Secretary and Treasurer be chosen from the 5 people up for nomination?

__Note:__ Now order matters! Joe could be Pres., Sue could be Secretary, Bill could be Treasurer, or a different permutation -- Sue could be Pres., Bill could be Secretary, Joe could be Treasurer, which is certainly different from the first arrangement.

Example: Let's put the concepts together with probability for one last example.

I give homework assignments to my Algebra classes where I randomly choose 5 problems to grade.

a) If there are 18 problems, how many ways are there to choose 5 from among the 18?

b) If a student only did 5 problems out of 18, what is the probability that the student did the 5 problems that I chose to correct?

__Note:__ This is the probability that just 1 grouping of all those possible is chosen!

c) If a student did 12 of the 18 problems, how many different groups of 5 could this student have done?

__Note:__ This is asking, how many groups of 5 are in 12!

d) What is the probability that the student that did 12 problems, did the 5 problems that I graded?

__Note:__ This student has many more chances of the correct grouping of 5 being chosen out of the ${}_{18}C_5$ *possibilities!*

§5.2 Random Variables, Probability Histograms and Probability Distributions

A random variable (r.v.) can be either continuous or discrete. It takes on the possible values of an experiment. It is usually denoted: x when discussing values; X when describing the outcomes

Example: a) What are the values x can take on for the roll of a single die? Is this a discrete or continuous r.v.?

b) What are the values x can take on for the altitude of an airplane that takes off from San Francisco airport, assuming that it does not crash? Is this a discrete or continuous r.v.?

c) Suppose a coin is tossed twice so that the sample space, S, is the following set {HH, HT, TT, TH}. Let X represent the number of heads that come up. What are the possible values of the r.v. x?

A probability distribution is a function that describes the probability associated with each value of a random variable.

Example: Describe the probability distribution associated with the rolls of a single die. In order to find the distribution you must first find the possible values of the r.v. x and then based upon classic probability you will find the probability of obtaining those values.

****Note:** There is a functional relationship between the values of a r.v. and the probabilities associated with them.

Example: Describe the probability distribution associated with the number of heads obtained when a coin is tossed twice.

We should now discuss the fact that associated with every probability distribution there are certain rules which must be adhered to. They are as follows:

1. $\sum P(x) = 1$

2. $0 \leq P(x) \leq 1$

Example: a) Notice in the rolled die experiment where X = # on the die that the $\sum P(x) = 1$ and that $0 \leq P(x) \leq 1$

b) Show the same holds true for the coin example

Example: Determine if the following is a probability distribution and indicate which of the rules have been violated if it is not a pdf.

a)

x	1	2	3
f(x)	¼	¼	¾

b)

x	0	1	2
f(x)	- ¼	1	0

A probability histogram is a special case of a relative frequency histogram. It shows each probability as a rectangle whose area is equivalent to the probability of the random variable's value. As such the area under the "curve" is always equal to one. The following is a summary of the characteristics of a probability histogram.

1. Every bar is centered over a value of a random variable.
2. Every bar is one unit wide.
3. Every bar touches the one next to it.
4. The height of a bar is equivalent to the probability of the r.v.'s value
5. The area in each bar is equivalent to the probability (2 & 4 multiplied)
6. The sum of the areas under each bar is always 1. Said another way the area under the "curve" is always one.

Example: a) Draw a probability histogram for the die rolling example.

b) Draw a probability histogram for the coin toss example.

From a probability distribution we can see 3 characteristics of the data:

1) Shape – Via the histogram
2) Mean – By calculation or by the histogram
3) Variance – By calculation or by the histogram

Finding the Mean Using the PDF

$\mu = \sum x \cdot P(x)$

Finding the Variance Using the PDF

$\sigma^2 = [\sum x^2 \cdot P(x)] - \mu^2$