## §9.3 & 9.4 Simplify Radical Exp. & Solving Quadratics Using Sq. Rt. Property

*The square root "undoes" a square. It gives us the base when the exponent is 2!*

**Example:** $\sqrt{4} \Rightarrow (?)^2 = 4$

The ? is the square root of 4. It is the base that when raised to the second power will yield the radicand 4.

**Terms to Know:**

| | | |
|---|---|---|
| $\sqrt{\ }$ | - | **Radical Sign** |
| $\sqrt{4}$ | - | The 4 is called the **radicand** |
| $\sqrt{4}$ | - | The entire thing is called a **radical expression** (or just radical) |

A square root has both a positive and a negative root. The positive square root is called the **principle square root** and when we think of the square root this is generally the root that we think about, but both a positive and negative square root exist. Your book, when it asks for the square root of a number is asking for the principle square root, but I think that it is best if you put yourself in the habit of thinking of both the positive and negative roots to a square root!

**Good Idea to Memorize**

$1^2 = 1$; $2^2 = 4$; $3^2 = 9$; $4^2 = 16$; $5^2 = 25$; $6^2 = 36$; $7^2 = 49$; $8^2 = 64$; $9^2 = 81$; $10^2 = 100$; $11^2 = 121$; $12^2 = 144$; $13^2 = 169$; $14^2 = 196$; $15^2 = 225$; $16^2 = 256$; $25^2 = 625$

***Note:*** *There is no real number that is the square root of a negative number, and so your answer to such a question at this time would be:* ***not a real number****.*

**Example:** Find the root of the following.

a) $\sqrt{256}$ b) $-\sqrt{144}$ c) $\sqrt{-625}$

***Note for c):*** *The negative square root has a solution that is a real number, but the square root of a negative does not!*

d) $\sqrt{1/16}$ e) $\sqrt{4/9}$ f) $\sqrt{0}$

There also other roots and these are shown with the radical sign and an index. The index is a superscript number written just outside and to the left of the radical sign. The index does not appear in the square root because it is assumed to be 2 (the square root) if nothing is written.

**More to Know**

| | | |
|---|---|---|
| $\sqrt[3]{\ }$ | : | This is the **cube root**; 3 is the **index** |
| $\sqrt[4]{\ }$ | : | This is the **4th root**; 4 is the **index** |

The cube root "undoes" a cube. The fourth root gives us the number that when raised to the fourth power will give us the radicand. A good way to think of the 4th power is the square of the square!! Or to undo it the square root of the square root.

**Good to Memorize Too**

$1^3 = 1$; $2^3 = 8$; $3^3 = 27$; $4^3 = 64$; $5^3 = 125$

***Note:*** *The cube root or any odd indexed root can have the a real solution when the radicand is negative, since a negative times a negative times a negative, etc., is a negative number. Whenever the index is even, it is not possible to have a real number solution to a negative radicand!*

**Example:** Find the cube roots of the following:

a) $\sqrt[3]{8}$ b) $\sqrt[3]{-1/125}$ c) $-\sqrt[3]{-27}$

***Note:*** *I expect you to know that $2^5=32$*

**Example:** $\sqrt[4]{256}$

***Note:*** *It may be helpful to think of the $4^{th}$ root as the square root of the square root, since you can easily do the square root of 256 and then you can again take the square root of that answer. All indexes can be broken down in this way by thinking of the product of the indexes (using the exponent rule for raising a power to a power). The sixth root could be thought of as taking the square root and then the cube root or vice versa depending upon which will get you better results! This is a legal operation, because roots can be written as exponents that are the index's reciprocal, and the radicand is the base.*

**Example:** $\sqrt[4]{-16}$

***Recall:*** *If the index is even there is no real solution when the radicand is negative!*

Radicals can be rewritten as **rational exponents** and simplified using prime factorization & the following definition of rational exponents.

**Rational Exponents**

$$\sqrt[n]{a^m} = (a)^{m/n}$$

Let's look at how that can help us with the $4^{th}$ root of 256. Once you have used prime factorization, you can see the problem as being asked in the following manner:

$$(2^{?})^4 = 2^8$$

**Approximation to Irrational when Radicand Not a Perfect Square (Cube, etc.)**

When the radicand is not a perfect square, perfect cube, etc., our book suggests that we use an approximation to the irrational number at this time. The best way to find an approximation is to use your calculator! If you do not have a calculator then the best that you can do is to give an approximate value for the actual value by looking at the 2 perfect squares which bracket the answer or by using a table.

**Example:** $\sqrt{8}$

## Exact Value of an Irrational Root

Now we can learn what to do with the square root of a number that is not a perfect square (or the cube root that is not a perfect cube, etc.). Instead of saying it is approximately such and such, we will give a much more satisfying answer in terms of a whole number multiplied by a radical. This will allow us to do many things with radicals!

A radical is considered to be in **simplified form** when as much as possible has been removed from under the radical sign. There are 2 properties that will be used to achieve an exact value. I will give them in terms of the square root, but you must understand that they extend in concept to any higher indexed radical as well, even though I will not explicitly state those extensions.

**Product Rule of Square Roots**

$\sqrt{xy} = \sqrt{x} \bullet \sqrt{y}$ $\qquad\qquad$ $x, y \in \Re \geq 0$

**Quotient Rule of Square Roots**

$\sqrt{x/y} = \sqrt{x}/\sqrt{y}$ $\qquad\qquad$ $x \in \Re \geq 0$ and $y \in \Re > 0$

*Remember that a denominator can't be equal to zero or the problem is undefined!*

We will be using the product rule by rewriting radicands into products, where one factor is a perfect square (or cube, or whatever the index). Here again is a place where it is quite helpful to know your multiplication tables quite thoroughly!!

**Steps in Simplifying a Radical Using Product Rule**

1. Find the largest perfect square that is a factor (or cube, etc.), and rewrite the radicand as a product.
2. Use the product rule to rewrite the radical.
3. Simplify to a number (the root of the perfect) multiplied by the radical of the other factor.

***Note:** In place of steps 1&2 prime factorization and the fact that $\sqrt[n]{a^m} = a^{m/n}$ can be used instead.

**Example:** Simplify

a) $\sqrt{8}$ $\qquad$ b) $\sqrt[3]{-54}$ $\qquad$ c) $\sqrt{16}$

**<u>Steps for Simplifying Radicals Using the Quotient Rule</u>**

1. Use the quotient rule to separate the problem into a numerator and a denominator.
2. Follow the steps for the product rule on both the numerator and the denominator.
3. Cancel if necessary, to further simplify.
4. *Although it will not come up in this section, there is another step that is called rationalizing the denominator. This must occur when the denominator still contains a radical. Having a radical in the denominator is not considered simplified!! We will learn how to deal with this case in section 4.*

**Example:** Simplify

a) $\sqrt[3]{15/64}$ b) $-\sqrt{27/16}$ c) $\sqrt[4]{32/81}$

But, what if part c) was $\sqrt[4]{81/32}$? Well that isn't allowed! A radical expression is NOT considered SIMPLIFIED if there is **a radical in the denominator**. If there is a radical in a denominator then a process called rationalizing must occur. Rationalizing a denominator means multiplying the denominator and the numerator (so that an equivalent fraction is maintained) by a root such that will create a perfect square, cube, etc. Since we should already be in the habit of using prime factors we shouldn't have too difficult of time making perfect squares, cubes, etc, as that is just multiplying by another factor with an exponent that equals the difference between the perfect square (cube, etc.) and the current exponent.

**Example:** $$\frac{1}{\sqrt{8}} = \frac{1}{\sqrt{2^3}} \bullet \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2^4}} = \frac{\sqrt{2}}{2^2} = \frac{\sqrt{2}}{4}$$

↑Next perfect square is 4.

↑Mult. by 2 to the exp. of 4 – 3

↑Creating a perfect square

**Example:** $$\frac{1}{\sqrt[3]{12}} = \frac{1}{\sqrt[3]{2^2\bullet 3}} \bullet \frac{\sqrt[3]{2\bullet 3^2}}{\sqrt[3]{2\bullet 3^2}} = \frac{\sqrt[3]{18}}{\sqrt[3]{2^3\bullet 3^3}} = \frac{\sqrt[3]{18}}{2\bullet 3}$$

$$= \frac{\sqrt[3]{18}}{6}$$

**Your Turn**

**Example:** Simplify. Assume that all variables represent positive #s.

a) $\dfrac{2}{\sqrt{7}}$ b) $\dfrac{5}{\sqrt[3]{2}}$

c) $\sqrt{\dfrac{2}{27}}$ d) $\dfrac{\sqrt{2}}{\sqrt{14}}$

Our next task is to learn to simplify a rational expression containing a radical expression in the numerator. We will be using this process when we apply the square root property to solve quadratic equations in the next section.

**Steps to Simplification**

1) Simplify any radical in the expression completely
2) Factor GCF from any polynomial (probably a binomial, but I'm being general)
3) Cancel common factors in the numerator & denominator (only after factoring though)
4) Rewrite

**Example:** $\dfrac{4 + \sqrt{8}}{4} = \dfrac{4 + 2\sqrt{2}}{4} = \dfrac{\cancel{2}(2 + \sqrt{2})}{\cancel{4}} = \dfrac{2 + \sqrt{2}}{2}$

↑ Notice the 2's can't cancel because there is addition

**Your Turn:**

a) $\dfrac{21 - 7\sqrt{11}}{14}$ b) $\dfrac{12 + \sqrt{72}}{18}$

One more new piece of knowledge before we put it altogether:

## Complex Numbers

First we need to know about the **imaginary unit** *i*.

$$i^2 = -1 \text{ and therefore } \sqrt{-1} = i$$

***Note:*** *You will notice that the book takes everything to $\sqrt{-1} = i$, but I go with $i^2$ under the radical to represent the –1. I find this less labor intensive as $i^2$'s root is i, giving the same end result.*

What this really means is that we can now take the root (when the index is even) of a negative number. Roots of negative numbers are called **complex numbers**. Complex numbers are numbers that have an imaginary component.

$$a + bi \qquad\qquad a,b \in \Re \text{ and "}i\text{" is the imaginary unit}$$

Our first step is to write complex numbers in the form a + bi. This requires the examples above applied to negative numbers under radicals with even indexes and then a simplification.

**Example:** Simplify the following to the form a + b*i*

a) $7 - \sqrt{-16}$ b) $-1 + \sqrt{-8}$ c) $\sqrt{-72}$

***Note:*** *When there is no a component of a complex number the a is zero, hence 0 + bi.*

Now, we will combine our concepts and use them to solve quadratic equations. I already mentioned this when we were finding the x-intercepts of a parabola in section 9.1 with the vertex form of a parabola.

## The Square Root Property

$$\text{If } \quad x^2 = a \text{ and } a \in \Re, \text{ then } \quad x = \pm\sqrt{a}$$

**Note: The square root of any number yields both a + and a – root since +•+ = + and – • – = +!!*

## Solving a Quadratic with the Square Root Property

***Note**: Method can't be used if there is a $1^{st}$ degree 'x' term

1. Get the $x^2$ term [this term could be complex and look like $(x + \#)^2$]alone on one side and the constant on the other. Remove any coefficient of $x^2$ by division.
2. Take the square root of both sides remembering that the square root of any number can be both + or –. Also remember that there is no real number that equals the square root of a negative number.
3. If the $x^2$ term was complex, move the remaining constant from the left to the right side of the equation using the addition property.

**Examples:** Find the solution to

a) $p^2 = {}^1/_{49}$ b) $(m - ½)^2 = ¼$ *c) $(z - 4)^2 = -9$

*To finish this problem we will need the skill of dealing with complex numbers.

**Your Turn:** Solve using the square root property.

a) $x^2 - 144 = 0$  b) $x^2 + 144 = 0$  c) $(x + 1)^2 = 12$

d) Find the x-intercepts of the parabola $f(x) = 2(x + 1)^2 - 4$

**Class Exercises & Homework**

§9.3 p. 508 #8, 9, 12, 13, 14, 27, 32, 45, 46, 47, 51, 54, 61, 64, 71, 74, 80, 88, 91, 98, 99

§9.4 p. 517 #8, 9, 12, 18, 25, 28, 31, 42, 43, 46, 51, 62, 66, 67, #71-73, 77, 92, 102, 103