

Ch. 12 Rational Functions

§12.1 Finding the Domains of Rational F(n) & Reducing Rational Expressions

Outline

Review

Rational Numbers – $\{^a/b \mid a \text{ and } b \text{ are integers, } b \neq 0\}$

Multiplying a rational number by a fraction of 1 doesn't change the rational number

i.e. Fractions multiplied by $^a/a$ are equivalent fractions

Rational Expressions – $\{^P/Q \mid P \text{ \& } Q \text{ are polynomials, } Q \neq 0\}$

Properties – *Just as in any fraction when the numerator & denominator are multiplied/divided by the same #, it results in an equivalent statement*

Restrictions – *Because the denominator can't equal zero we must find the zeros for the denominator polynomial, and these are called the restriction*

How – Set denominator = 0 and solve for variable (using quadratic or linear equation skills)

Reducing – *Just as with any fraction we will be using canceling of like terms (i.e. division, which is a property that is valid to use with rational expressions)*

How – Factor the numerator and denominator and cancel like terms (i.e. constants or polynomials)

Evaluation – *Caution in evaluating the simplified form of a rational expression, pay attention to restrictions*

Function Notation – *Evaluation via function notation*

Recall that a rational number is a quotient of integers. A rational expression is a quotient of polynomials, such as:

$$\frac{P}{Q} \quad \text{where } Q \neq 0 \text{ and } P \text{ and } Q \text{ are polynomials}$$

A rational function is a function whereby a rational expression is used to obtain the values of the function.

A rational function can be evaluated just as any function, **except** that a rational expression can be **undefined when the denominator is equal to zero.**

Example: For $f(x) = \frac{x^2 - 3}{x + 9}$ find $f(5)$

Example: For $f(x) = \frac{x^2 - 3}{x + 9}$ find $f(-9)$

The numbers for which a rational function are undefined are called the restrictions on the domain of the rational function. In order to find the restrictions we must find the points that the rational function's denominator is equal to zero. We do this by setting the denominator equal to zero and solving the resulting equation. The answer(s) are the restrictions of the rational function's domain.

Finding the Restrictions of a Rational Function

Step 1: Set the denominator equal to zero

Step 2: a) Solve the resulting quadratic equation using the zero factor property

b) Solve the resulting linear equation by applying the addition and multiplication properties of equality.

Step 3: The restrictions are the roots of the quadratic solved in step 2 **or** the solution to the linear equation in step 2. Write them as $x \neq \underline{\hspace{2cm}}$ using set builder notation.

Example: Using set builder notation give the domain of each of the following.

a) $f(x) = \frac{x^2 + 2x + 1}{x + 1}$

b) $f(x) = \frac{x^2 - 3}{x^2 - 3x - 10}$

Example: Are there any restriction on the domain of

$$f(x) = \frac{x^2 + 3x - 10}{x^2 + 5} ?$$

Recall that

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \text{ when } b \neq 0$$

This is also true of polynomials. My suggestion is to simply pull any negative signs out and consider the entire expression as either a positive or negative expression.

Example: $\frac{(3 - x)}{x^2 - 3x + 2} = -\frac{x - 3}{x^2 - 3x + 2}$

Just as when dealing with fractions, if the numerator and denominator are multiplied by the same thing the resulting expression is equivalent. This is called the Fundamental Principle of Rational Expressions, when we are discussing a fraction of polynomials (a rational expression).

$$\frac{P \cdot R}{Q \cdot R} = \frac{P}{Q} \quad \text{if } P, Q \text{ and } R \text{ are polynomials and } Q \neq 0$$

Concept Example: $\frac{15}{35} = \frac{3 \cdot \cancel{5}_1}{7 \cdot \cancel{5}_1} = \frac{3}{7}$

In order to simplify rational expressions we will use the Fundamental Principle of Rational Expressions just as we used the Fundamental Principle of Fractions to simplify fractions.

Simplifying a Rational Expression

- Step 1:** Find the any restrictions on the rational expression (as above)
- Step 2:** Factor the numerator and the denominator completely
- Step 3:** Cancel common factors
- Step 4:** Rewrite

Concept Example: Simplify $\frac{15xy}{35x}$

Note: Another way of applying this principle is division!

Example: Simplify each of the following. Don't forget to find the restrictions first.

a) $\frac{x^2}{x^2 + 2x}$ b) $\frac{x + 5}{x^2 + 2x - 15}$

c) $\frac{x^2 - x - 2}{x^2 + 5x - 14}$ d) $\frac{-x + y}{x - y}$

e) $\frac{x - 2}{x^2 - 4}$

Example: Why is it when we evaluate $\frac{x - 2}{x^2 - 4}$ and $\frac{1}{x + 2}$,

for $x = 2$, we don't get the same number, although we know that they are equivalent expressions (see last example)?

Your Turn §12.1

1. Give the restrictions & simplify each of the following.

a) $\frac{x^2 + 4}{x^2 - 5x - 14}$

b) $\frac{-x^2 - 3x - 2}{x^2 - x - 2}$

2. Simplify $\frac{6x^2 + 9xy - 2xy - 3y^2}{2x^2 - 2x - 3xy + 3y}$

3. Find $f(-1)$ for a) in #1

§12.2 Multiplying and Division of Rational Expressions

Outline

Multiplying (Dividing—Mult. by reciprocal) Rational Expressions

Just like fractions

- How –
- 1) Simplify numerator & denominators using factoring
 - 2) Cancel like terms
 - 3) Multiply numerators
 - 4) Multiply denominators

Unit Analysis

Write problems as a multiplication of fractions in terms of the units with current units in numerator and final units in numerator on right of equal sign.

Write ratio of desired units to current units or several ratios that will achieve the desired conversion.

Remember that the goal is to cancel the undesired units and end up with the new units.

Multiplying rational expressions is just like the 1st process that you learned for multiplying fractions. You may not remember that process, but it went like this: 1) Factor the numerator and denominator into their prime factors 2) Cancel like terms 3) Multiply numerators 4) Multiply denominators (Remember that if it is a whole number to place it over one)

Concept Example:
$$\frac{2}{3} \cdot \frac{3}{8} = \frac{\cancel{2}^1}{\cancel{3}_1} \cdot \frac{\cancel{3}^1}{\cancel{2}_1 \cdot 2 \cdot 2} = \frac{1}{4}$$

Concept Example:
$$21 \cdot \frac{7}{30} = \frac{\cancel{3}^1 \cdot 7}{1} \cdot \frac{7}{2 \cdot \cancel{3}_1 \cdot 5} = \frac{49}{10}$$

Note: With fractions we would have had one final step – to convert this to a mixed number, but this does not apply to rational expressions, so this is where our concept example ends.

Multiplying Rational Expressions

Step 1: Factor numerators and denominators completely

Step 2: Cancel if possible

Step 3: Multiply numerators

Step 4: Multiply denominators

Example: Multiply

a)
$$\frac{x+2}{x^2+7x+6} \cdot \frac{x+1}{x^3+8}$$

b)
$$\frac{x^2}{x^2+2x+1} \cdot \frac{x+1}{x^2-x}$$

$$c) \quad (x^2 - 25) \cdot \frac{x - 1}{2x^2 - 11x + 5}$$

$$d) \quad \frac{x}{3 + x - 2x^2} \cdot (x + 1)(2x - 3)$$

Just like multiplication, division of rational expressions takes its cue from fractions. We multiply by the reciprocal of the divisor and then follow the steps from above for multiplication.

Concept Example: $\frac{1}{2} \div \frac{2}{3} = \frac{1}{2} \cdot \frac{3}{2} = \frac{3}{4}$

Dividing Rational Expressions

Step 1: Take the reciprocal of the divisor (that's the 2nd expression).

Step 2: Multiply the dividend (that's the 1st expression) and the reciprocal of the divisor.

Example: Divide each of the following.

$$a) \quad \frac{x + 1}{2x^2 - x + 1} \div \frac{3x^2 - 3}{x^2 - 3x + 2}$$

$$b) \quad \frac{4x^2 - 16}{x + 1} \div \frac{8x^2 - 32}{x^2 + 9x + 8}$$

$$c) \quad \frac{2x^2 - 4xy - xy + 2y^2}{5y + 35} \div \frac{2x^2 + xy - y^2}{y^2 + 6y - 7}$$

Last, a formal discussion of unit analysis. Unit analysis is the formation of an expression based upon the desired end result of the units. It is achieved by starting with the current units and using canceling to arrive at the desired end units. It is a very difficult process to put in concise terms, but I will give it a try.

Unit Analysis

Step 1: Write the current units as a fraction.

Step 2: Write the final units as a fraction on the right side of an equal sign.

Step 3: Using this step one or more times proceed to the final desired units.

- a) Find a conversion that applies to the unit conversion that you are attempting to make, and write it as a fraction such that your current units are in the denominator/numerator (such that they are opposite those that currently exist) and will therefore be canceled.
- b) Continue to find applicable conversions until the units in the numerator/denominator are those on the right side of the equal sign.

Step 4: Put the appropriate numbers with their units and follow the directions given by unit equation.

Example: Convert 27 inches to feet using unit analysis.

Example: Convert 172 inches to yards using two step 3's in the outlined method for unit analysis.

Example: Convert 90 mph to miles per minute.

Example: Convert 90 mph to feet per second. (5,280 ft. per mile)

Your Turn §12.2

1. Multiply.

$$\text{a) } \frac{x^2 - 5x - 14}{x^2 - 1} \cdot \frac{x^2 - 8x + 7}{x^2 + x - 2}$$

$$\text{b) } \frac{15x^2 - 25x - 3x + 5}{5(5 - 3x)} \cdot \frac{5x^2 - 20}{2 - 9x - 5x^2}$$

2. Divide.

$$\text{a) } \frac{x^2 + 8}{x^2 - 16} \div \frac{x^3 - 10}{x^2 - 9x + 2}$$

$$\text{b) } \frac{6x^2 - 3x - 3}{9x^2 + 12x + 4} \div \frac{3(x - 1)}{10 - 15x}$$

$$\text{c) } \frac{5x^2 - x + 5}{5x^2 - 6x + 1} \div \frac{5x - 1}{x^2 - 2x + 1}$$

3. Use unit analysis to convert each of the following.

a) How many inches is 7 feet?

b) How many inches is 2 miles?

c) How many yards per second is 18 miles per minute?

§12.3 Adding and Subtracting Rational Expressions

Outline

Adding/Subtracting w/ Common Denominators

Add/Subtract Numerators being cautious to distribute subtraction

Factor and cancel just as with fractions

Adding/Subtracting w/ Unlike Denominators

LCD – Product of primes (this time a prime is considered a factor as well as a prime number)

Find the LCD, Build the higher term, Multiply out numerators, Add/Subtract Numerators, Factor & Cancel

The Sum of a Number and Its Reciprocal

Preparation for Applications

Denote number by x and therefore its reciprocal is $1/x$

As in the last section we will be taking our cues for adding/subtracting rational expressions from fractions. When we have a common denominator, as with fractions, we simply add or subtract the numerators. We do have to be cautious because when subtracting it is the entire numerator that's subtracted, so we must use the distributive property to subtract.

Concept Example: Add or subtract

$$\text{a) } \frac{7}{15} - \frac{3}{15} = \frac{4}{15}$$

$$\text{b) } \frac{7}{16} + \frac{5}{16} = \frac{12}{16} = \frac{\cancel{2}^1 \cdot \cancel{2}^1 \cdot 3}{\cancel{2}_1 \cdot \cancel{2}_1 \cdot 2 \cdot 2} = \frac{3}{4}$$

Adding/Subtracting Rational Expressions w/ Like Denominators

Step 1: Make sure that numerators are in expanded form and denominators are factored.

Step 2: Distribute the subtraction if subtracting.

Step 3: Add the numerators by combining like terms.

Step 4: Factor the resulting numerator and the denominator if that was not done in 1st step.

Step 5: Cancel like terms and rewrite.

Example: Add or Subtract

$$\text{a) } \frac{5}{y} + \frac{-3}{y}$$

$$\text{b) } \frac{2x + 5}{y + 2} - \frac{x - 5}{y + 2}$$

c)
$$\frac{2}{y-2} - \frac{y}{y-2}$$

d)
$$\frac{x^2 + 1}{x + 1} + \frac{2x}{x + 1}$$

Note: We always need to factor the numerator and denominator in order to reduce and also for our next application of finding a common denominator.

e)
$$\frac{(x-1)^2}{x+1} + \frac{2x}{x+1}$$

Finding the LCD of a rational expression is the same as finding the LCD of fractions. We just need to remember that a polynomial must first be factored. Each factor that isn't a constant is considered like a prime number.

Concept Example: Find the LCD of $\frac{2}{15}$, $\frac{7}{36}$

Note: The LCD of $15 = 3 \cdot 5$ and $36 = 2^2 \cdot 3^2$ is $180 = 2^2 \cdot 3^2 \cdot 5$, which is the highest power of each unique factor.

Finding the LCD for Rational Expressions

Step 1: Factor the denominators (equivalent to factoring into primes for fractions) [Don't forget that numbers still need to be factored into their primes!]

Step 2: A unique factor to the highest power is always a factor in the LCD, so find the highest power of every unique factor and multiply them.

Example: Find the LCD of

a) $\frac{1}{x}$, $\frac{2x + 5}{x(x + 5)}$

b) $\frac{x + 2}{x^2 + 3x + 2}$, $\frac{x + 1}{x + 2}$

Next, we need to build a higher term so that we can add and subtract rational expressions with unlike denominators. Again our cues are taken from fractions

Concept Example: Build the higher term $\frac{2}{15}$, $\frac{7}{36}$

1) LCD was 180

2) $180 \div 15 =$ $180 \div 36 =$

or $3 \cdot 5 \cdot ? = 2^2 \cdot 3^2 \cdot 5$ $2^2 \cdot 3^2 \cdot ? = 2^2 \cdot 3^2 \cdot 5$

3) $\frac{2}{15} \cdot \frac{\quad}{\quad} = \frac{\quad}{180}$ $\frac{7}{36} \cdot \frac{\quad}{\quad} = \frac{\quad}{180}$

Building A Higher Term

Step 1: Find LCD

Step 2: Divide denominator by LCD (or ask yourself, "What multiplied by the current denominator will give the LCD?" Or, "What is the factor(s) that are missing?")

Step 3: Multiply numerator by quotient from step 2.

Example: Build the higher term

a) $\frac{1}{x}, \frac{2x + 5}{x(x + 5)}$

Recall: LCD was $x(x + 5)$

b) $\frac{x + 2}{x^2 + 3x + 2}, \frac{x + 1}{x + 2}$

Recall: LCD was $(x + 2)(x + 1)$

Addition and Subtraction of Rational Expressions w/ Unlike Denominators

Step 1: Find LCD (see steps from page 12 of notes)

Step 2: Build Higher Term (see steps from page 13 of notes)

Step 3: Add/Subtract as normal (see steps from page 10-11 of notes)

Example: Add/Subtract each of the following

a) $\frac{15a}{b} + \frac{6b}{5}$

b) $\frac{13}{2x - 4} + \frac{x}{x^2 - 4}$

c) $\frac{x}{x^2 - 4} - \frac{2}{4 - x^2}$

d) $\frac{y + 2}{y + 3} - 2$

e) $\frac{x}{x^2 - 1} - \frac{2}{x^2 - 2x + 1}$

Last in this section is an application of rational expressions that is usually seen in a section on applications of rational equations. We will need to recall the meaning of a reciprocal and **focus on translation!** In the following problems we will define the variable and then translate the expression involving the variable which may involve the reciprocal.

Example: Write an expression for the difference of a number and its reciprocal. Don't forget to simplify if possible.

Example: Write an expression for the sum of a number and the reciprocal of twice that number.

Example: Write an expression for the difference of twice a number and twice the reciprocal of the number. Don't forget to simplify.

Your Turn §12.3

1. Find LCD of

a) $\frac{2x}{x^2 + 2x + 1}$, $\frac{x^2 + 1}{x^2 - 1}$

b) $\frac{2}{4x^2 + 13x + 3}$, $\frac{x + 1}{2x^2 + 12x + 18}$

2. Build the higher term for

a) $\frac{2x}{x^2 + 2x + 1}, \frac{x^2 + 1}{x^2 - 1}$

b) $\frac{2}{4x^2 + 13x + 3}, \frac{x + 1}{2x^2 + 12x + 18}$

3. Add/Subtract

a) $\frac{2x}{x^2 + 2x + 1} + \frac{x^2 + 1}{x^2 - 1}$

b) $\frac{2}{4x^2 + 13x + 3} + \frac{x + 1}{2x^2 + 12x + 18}$

c) $\frac{x}{x + 3} - \frac{3}{x + 3}$

d) $\frac{2y}{2(3 - y)} - \frac{6}{2(3 - y)}$

4. Write an expression for the sum of one more than three times a number and three times the number's reciprocal. Simplify if possible.

HW for §12.1-12.3

§12.1 p. 706 #15, 22, 54, #60-64all #67, 68, 71 & 74

§12.2 p. 716 #26, 45, 48, 50, 55, 58, 63, 73, 78, 79, & 81

§12.3 p. 729 #4, 5, 12, 13, 16, 21, 27, 36, 41, 46, 50, 51, 54, 59 & 83