§11.2 Logarithmic Functions

A logarithmic function is the inverse function of an exponential function. If you recall from §9.1, a functions inverse can be found by exchanging x and y and solving for y. At this point we can exchange x and y but we have no basis for solving for y. This is where the definition of a logarithm comes in:

$$\mathbf{y = \log_a x \quad means \quad x = a^y \qquad a > 0 \ \& \ a \neq 1}$$

Now we have a way to solve for y, when we exchange the values of x and y in an exponential function. By the way, log, is an abbreviation for logarithmic. The above is read, y is the log of x to the base a. If by some off chance you have encountered a log before, you may never have seen the subscript a and that is because those were a common log function known as log base 10.

Here's the long and short of it: The answer, y, is the **exponent** of the base a, that gives us x. Here is an example to help you see this:

Example: $10^2 = 100$ so $2 = \log_{10} 100$

Example: Give the logarithmic form of each equation.

a) $2^5 = 32$ b) $(^1/_2)^3 = {}^1/_8$

c) Give a second equivalent log for b)

Hint: *Think of another way to write ½, with the exponent.*

Example: Give the exponential form of each log

a) $\log_2 16 = 4$ b) $\log_{10} 1000 = 3$

c) $\log_3 27 = 3$

Note: *I try to train my eyes to look to the subscript raised to the answers power to equal the middle number. Or simply use the idea of the inverse.*

Example: Find the unknown. It may help you to write the argument in exponential form first.

a) $\log_7 49 = y$

b) $\log_3 81 = y$

c) $\log_4 64 = y$

d) $\log_2 64 = y$

Note: *Do you see a relationship between the answer for c & d? 4 is 2^2 so the exponent is 2 times the y for 4! $(2^2)^3 = 4^3$*

Solving a logarithmic function is solving an exponential function for the other variable since they are inverses of one another f(x) for a logarthim is finding y for an exponential and vice versa.

Example: For $f(x) = 3^x$

a) Find f(0)

b) Find $f^{-1}(3)$

c) Write b) as a logarithm

d) Create a table of values for x = 0, 1, 2 and show a) & b) in the table.

Example: For $f(x) = \log_2 x$

a) Find f(2)

b) Find $f^{-1}(0)$

c) Write b) as an exponential

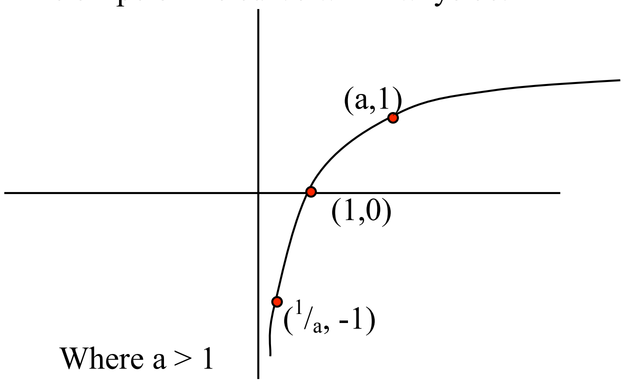
d) Create a table of values for x = 1, 2, 4 and show a) & b) in the table.

Graphing a Log Function

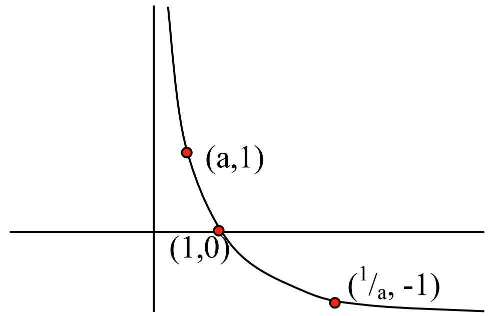
1) Change to exponential form
2) The domain is $(0, \infty)$.
3) The range is $(-\infty,\infty)$.
4) The points that are being plotted are $(^1/_a, -1)$, $(1,0)$ and $(a,1)$

Note: *The plot is the inverse of the exponential, the domains and ranges are flip-flopped! This is why all these key points seem so familiar!!*

The shape of the curve will always be:



Where a > 1
Increasing Function



Where 0< a < 1
Decreasing Function

Remember that log and exponential functions are inverses of one another. Thus their graphs are symmetric about the line $y = x$.

§11.2, 11.3 & 11.5 & 11.6 Log F(n) & Properties of Logarithms

Most of §11.2 and the other two sections are really methods for solving logarithmic equations. Since that is the case I'm going to cover them simultaneously and in the following manner.

Solving Log & Exponential Functions

First, let's recall our inverse functions. Our 1st task will be to switch between the two functions – logs to exponentials and exponentials to logs. I'm going to leave the §11.2 material in this section for you to review further.

$f(x) = a^x$ where $a > 0$, but $a \neq 1$
D: $(-\infty, \infty)$
R: $(0, \infty)$

$f(x) = \log_a x$ where $a > 0$, but $a \neq 1$
D: $(0, \infty)$
R: $(-\infty, \infty)$

We want to be able to switch seamlessly between the two inverse functions. This will help us in our equation solving skills later on. The key is that the domain of the exponential (the x) becomes the range of the logarithmic (the y), and the range of the exponential (the y) becomes the domain of the logarithmic (the x).

If $y = a^x$ then $\log_a y = x$ is the logarithmic equivalent.

§11.2 Example: $10^2 = 100$ becomes $\log 100 = 2$

Note: *There is no base written on the log in this case because the base 10 is what we call a common log and when you don't see a base written it is assumed to be base 10.*

You Try: Write the logarithmic equivalent for:
a) $5^2 = 25$ b) $2^{(x+1)} = 8$ c) $4^3 = z$

Of course we can always go the other way too.

§11.2 Example: $\log_3 x = 2$ becomes $3^2 = x$

You Try: Write the exponential equivalent for:
a) $\log_7 x = 3$ b) $\log_a 9 = 2$ c) $\log 10 = z$

I want to introduce another special base that Lehmann introduces in §11.6, and continue on. The next base is the natural log which is base *e*. This is another irrational number like π. It too is naturally occurring in natural processes having to do with growth and decay. Like π, it too has a common approximation. $e \approx 2.7183$. When the base of a logarithm is base *e*, we call it the natural log and it looks like: $\ln x = y$

§11.6 Example: Write the equivalent form:
a) $e^x = 1$ b) $\ln z = 5$

Finding a logarithm's value on your calculator can only be done if it is base 10 or base *e*. If we have a common log or a natural log it is a simple plug and chug calculation, but with any other base we will need something called the **base change formula**. This formula naturally arises from applying the fact that if a = b, then log a = log b and combining it with the property that $\log b^x = x \log b$. Lehman introduces this formula in §11.5.

$$\log_a b = \frac{\log b}{\log a} = \frac{\ln b}{\ln a}$$

Note: *That's log of argument divided by log of base.*

Example: Find the approximate value of the log to the nearest ten-thousandth.
a) $\log 5$ b) $\ln 2$

Example: Use the base change formula to find the exact value of the following and then approximate to the nearest ten-thousandth using your calculator.

a) $\log_2 5$ b) $\log_9 70$

You Try: Evaluate $\log_7 9$

Now, we want to investigate solving exponential and logarithmic equations using the a trick. The trick will be rewriting the exponential form so that the base is the same on each side of the equation. I introduced this idea in §11.2. This works because of the following property:

If $a^x = a^y$ (a is the same base), then $x = y$.

§11.2 Example: Rewrite each side so that you see like bases raised to the same power. Use the above property to find the value of the variable (you may need to solve an algebraic equation to accomplish the task).

a) $5^x = 5^2$ b) $3^x = 81$ c) $({}^1/_{16})^x = 2^4$

d) $2^{(x+1)} = 4^x$ e) $9^{(2x+3)} = 3^x$

Now, let's extend this concept to the logarithmic equations. We must first write the logarithmic equations in their exponential form, and then we can solve.

§11.2 Example: Rewrite in exponential form and then use the above property to solve.

a) $\log_2 4 = y$ b) $\log_{27} 3 = x$ c) $\log_3 {}^1/_{27} = z$

You Try: Using the principle that if the bases are the same then the exponents are equal, solve each of the following.

a) $9^x = {}^1/_{81}$ b) $\log_{81} 9 = x$ c) $\log 10{,}000 = y$

Now, another principle that can help us solve exponential and logarithmic equations. It again revolves around like bases, but it focuses on the logarithmic equation.

If $\log_a x = \log_a y$ (the bases are the same), then $x = y$ (note these are called the arguments).

Example: Solve the following. Make sure your solution is valid under the domain of the log function – $(0, \infty)$.

a) $\log (x + 1) = \log 2x$

You Try: $\ln (x - 1)^2 = \ln (3x - 5)$

The property is true in converse as well:

If $x = y$, then $\log_a x = \log_a y$

Example: Solve $2^{4x + 5} = 17$

You Try: Solve $8 = 3^{7x - 1}$

Two more related principles. This one is a little trickier. In this one you manipulate the equation by making the original the exponent.

$$a^{\log a\ (x)} = x$$

In other words, if the bases are the same then the argument of the logarithm in the exponent is the answer. This is useful when there is no possible way of rewriting the bases so that they are the same.

Example: Solve using this principle.

a) $\log_7 x = 2$

b) $\log_x 64 = 2$

You Try: $\log_9 x = {}^{1}/_{2}$

Now, the similar principle for exponentials. It requires taking the log of both sides and again revolves around the fact that the exponential can't be rewritten to have the same base. You may also need a calculator for this one!

$$\log_a a^x = x$$

Example: Solve the following.

a) $3^x = 4$ b) $10^{x+1} = 2$ c) $e^x = 1$

You Try: Solve $7^x = 20$

Next, we will learn some properties that will be used to condense/expand logarithmic expressions, which will then in turn be used to solve more complex logarithmic equations where the variable is in more than one argument. These properties are based on the product, quotient and power rules for exponents so they should look somewhat familiar to you!

§11.5Product Rule	**§11.5Quotient Rule**	**§11.3Power Rule**
$\log_a x + \log_a y = \log_a xy$	$\log_a x - \log_a y = \log_a \frac{x}{y}$	$y \log_a x = \log_a x^y$

Note: *$\log_a (x + y) \neq \log_a x + \log_a y$ nor does $\log_a (x - y) \neq \log_a x \quad \log_a y$*

Example: Use the above rules to condense the following:

a) $\log_2 x + \log_2 (x + 2)$ b) $\log_x 7 - \log_x 9$ c) $x \log_5 9$

You Try:

a) $\log_2 x - \log_2 (x + 2)$ b) $\log_7 x + \log_7 4$ c) $2 \log_9 x$

Now the opposite direction.

Example: Use the above rules to expand the following

a) $\log_x 2(x + 1)$ b) $\log_{17} 5^x$ c) $\log_2 ({}^4/_3)$

You Try:

a) $\log_7 {}^2/_{(x + 1)}$ b) $\log_y 5x$ c) $\log_z 4^3$

Altogether now…

Example: Condense the following using all properties that apply.

a) $2\log_2 x + 5\log_2 (x + 2)$

b) $2\log_3 x - \log_3 9 + \log_3$

Example: Expand the following using all properties that apply.

a) $\log_x 2(x + 1)^2$

b) $\log_{17} 5^x(x + 2)^3$

c) $\log_2 (^4/_3)x^2(x + 1)$

The reason that we learn these properties is so that we can use them to solve more complex logarithmic equations, such as the following.

Example: Solve the following: $\log x + \log (x + 3) = 1$

As you can see we are pulling all the pieces together.

1) Condense the equation
2) Variable is in the argument so we can use the comparable exponential form to get to an algebraic equation that we can use to solve for the variable.
3) Make sure that your solution is within the domain of the log function.

You Try:

a) $\log_3 x + \log_3 (2x - 3) = 2$

b) $\log_2 x + \log_2 (x - 7) = 3$

c) $\log_5 (x + 3) - \log_5 x = 2$

d) $3 \log x - 2 \log x = 2$

HW Problems

§11.2 p. 653 #2, 9, 12, 20, 21, 26, 29, 40, 45, 50, 53, 56, 61, 64, #69-72all, #77, 80 & 87

§11.3 p. 660 #1, 14, 24, 25, 30, 31, 34, 40, 41, 45, 49, 52, #56-64even, #71, 80,

§11.5 p. 678 #3, 6, 7, 12, 13, 18, 24, 26, 29, 38, 46, 54, 58, 62, 64 & 68

§11.6 p. 685 #4, 6, 9, 12, 14, 17, 20, 24, 30, 35, 38, 42, 44, 55, 58, 60 & 67