§15.1 Solving Equations & Inequalities with Absolute Values

Let's do some thinking about absolute values for a bit. $| a |$ means a without its sign, therefore the absolute value is always **greater than zero** if a is either larger or smaller than zero and if a = 0 then the absolute value is still **zero.** So, this leads us to the question:

Can **| a |** ever be negative (less than zero)?

Example: $| x | < 0$ has no $\Re$ solution since there is no number that will ever make this a true statement since the absolute is never less than zero.

Now, going back to the first paragraph and contemplate the meaning that the absolute value of any positive or negative integer is always positive. Think about the absolute value of opposites.

Is the value of **| a |** and **| -a |** the same?

Example: $| x | = 5$ so x can be either +5 or –5 since the absolute value of a number and its opposite are the same.

Now we need to give some consideration to | x | > #. There is one special case and that is when x is greater than a negative number.

What values will make | x | greater than any negative number?

Example: $| x | > -2$ this is all real numbers since no matter what number you put in the value is always ≥ zero

Now the | x | > the real numbers ≥ 0. This requires you to give a little consideration to the numbers on the number line and the fact that as you go out in each direction the number are ***getting larger in terms of their absolute value**. Also considered that in the negative direction as the numbers get larger in terms of their absolute value, they also get smaller in terms of their actual value, just as the positive numbers get larger. This gives us a hint as to how to solve the equations | x | > #.

Example: $| x | > 5$ since all positive numbers greater than 5 will make this true and all negative numbers less than –5 will make this true this is how we go about solving these problems.

Our last consideration is | x | < #, when that number is greater than zero. Again let's consider the positive numbers and the negative numbers that will make it a true statement. Positive numbers between zero and the number at hand will make a true statement. Negative numbers between zero and the opposite of the number at hand will also make the statement true, with much the same reasoning that we had above (*).

Example: $| x | < 5$ since all positive numbers less than 5 will make this a true and all negative numbers greater than –5 will make this a true statement, this is how we go about solving these problems.

Summary of Solutions

When **a** is a positive number

$\lvert x \rvert = a$	$x = a$ or $x = -a$	
$\lvert x \rvert < a$	$-a < x < a$	(x is between the + and – values of a)
$\lvert x \rvert > a$	$x > a$ or $x < -a$	(you may have seen this expressed in the following way, but you will not see me express it this way as it is not a correct statement, although it gets the job done. $-a > x > a$)

When **a** is a negative number

$\lvert x \rvert = a$	$\varnothing$
$\lvert x \rvert < a$	$\varnothing$
$\lvert x \rvert > a$	$\Re$

If **a** is zero we must be cautious of the endpoints!

$\lvert x \rvert > 0$	$\{x \mid x \in \Re, x \neq 0\}$*
$\lvert x \rvert \geq 0$	$\Re$
$\lvert x \rvert < 0$	$\varnothing$
$\lvert x \rvert \leq 0$	$x = 0$*

*x can be any algebraic expression and therefore you may have to solve to find the value of the variable!!

Solving an Absolute Value Equation

1) Get into $\lvert x \rvert > a$ **or** $\lvert x \rvert < a$ form by using addition/ multiplication properties
2) Set up appropriately to remove the absolute values
   a) If $\lvert x \rvert < a$ $\quad -a < x < a$
   b) If $\lvert x \rvert > a$ $\quad x > a$ or $x < -a$
3) Solve the resulting linear inequalities as usual
4) Check your solution

Very generally spoken!! For more detail see above.

Example: Find the solution set of the following

a) $\lvert x \rvert = 9$

b) $\lvert x \rvert = -7$

c) $\lvert c + 1 \rvert < 2$

d) $\lvert a - 2 \rvert + 3 < 4$

e) $|a - 2| + 3 < 2$

Note: *The absolute value can't be less than zero, so once you have completed step 1 of the process you know the answer.*

f) $\left| \dfrac{a + 3}{2} \right| \leq 0$

Note: *The absolute value can't be less than zero so you are only finding a single number that makes this statement equal to zero!*

g) $|c + 3| > 9$

h) $|d - 3| - 2 > 4$

i) $|e| + 2 \geq 0$

j) $|4 - 2y| \geq 0$

Note for j: *The absolute value is always greater than zero so you don't have to do anything to answer that the solution is all real numbers.*

k) $|16a - 4| > 0$

Note: *The endpoint is not included, so every real number except where $16a - 4 = 0$ will be the solution!*

Absolute Value Equals Absolute Value

In this case the value of one side equals both the positive and negative values of the other side. This too can be reasoned through by thinking about what it means for $|x| = |y|$. It means that whatever the value of x the value of y must also be the same, regardless of direction from zero, thus $x = y$ or $x = -y$.

Example: $|3r + 2| = |r - 3|$

Example: If $f(x) = |3x + 1|$ and $g(x) = |6x - 2|$, find all values for which $f(x) = g(x)$.